STEP 1 Mathematics 2019 Question 4 Algebra and Functions

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2019 Question 4
Algebra and Functions

4 **(i)** Find integers m and n such that $\sqrt{3+2\sqrt{2}} = m + n\sqrt{2}$.

(ii) Let $f(x) = x^4 - 10x^2 + 12x - 2$. Given that the equation $f(x) = 0$ has four real roots, explain why $f(x)$ can be written in the form

$$f(x) = (x^2 + sx + p)(x^2 - sx + q)$$

for some real constants s, p and q, and find three equations for s, p and q.

Show that

$$s^2(s^2 - 10)^2 + 8s^2 - 144 = 0$$

and find the three possible values of s^2.

Use the smallest of these values of s^2 to solve completely the equation $f(x) = 0$, simplifying your answers as far as you can.

$$3 + 2\sqrt{2} = m^2 + 2mn\sqrt{2} + 2n^2$$

$$3 = m^2 + 2n^2 \quad (1)$$

$$2 = 2mn \quad (2)$$

From (2), $m = \frac{1}{n}$, sub into (1),

$$\left(\frac{1}{n}\right)^2 + 2n^2 = 3$$

$$\frac{1}{n^2} + 2n^2 = 3$$

$$1 + 2n^4 = 3n^2$$

$$2n^4 - 3n^2 + 1 = 0 \qquad \text{let } n^2 = y$$

$$2y^2 - 3y + 1 = 0$$

$(2y-1)(y-1)=0$

$y=\frac{1}{2}$ OR $y=1$

$n^2=\frac{1}{4}$ $\quad$ $n^2=1$

$n=\frac{1}{4}$ OR $n=\frac{-1}{4}$ OR $n=1$ OR $n=-1$

$m=4$ OR $m=-4$ $\quad$ $m=1$ $\quad$ $m=-1$

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$\sqrt{3+2\sqrt{2}} = 2+\frac{1}{2}\sqrt{2}$ (rej), $m,n \in \mathbb{Z}$

$-2-\frac{1}{2}\sqrt{2}$ (rej) $m,n>0$

$1+\sqrt{2}$ ✓

$-1-\sqrt{2}$ (rej) $m,n>0$.

$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$

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$x^4 - 10x^2 + 12x - 2 = f(x)$

$x^4 + 0x^3 - 10x^2 + 12x - 2 = f(x)$

$(x^2+sx+p)(x^2-sx+q)$: x^3 term.

$sx^3 - sx^3 = 0x^3$ $\Rightarrow$ also roots are in conjugates

coefficient of x^3 term is zero.

$px^2 + qx^2 - s^2x^2 = -10$ x^2

$p + q - s^2 = -10$ — 1

$qsx - psx = 12x$ x

$qs - ps = 12$

$s(q-p) = 12 \quad -(2)$

$pq = -2 \quad -(3)$

$\times 0$

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$p + q - s^2 = -10 \quad -(1)$

$(q-p)s = 12 \quad -(2)$

$pq = -2 \quad (3)$

$\boxed{\text{Eliminate } p, q}$

$p + q = s^2 - 10$

$(p+q)^2 = (s^2-10)^2$

$$q-p=\frac{12}{s}$$

$$(p\pm q)^2=p^2+q^2\pm 2pq$$

$$(p-q)^2=\frac{144}{s^2}$$

$$(p+q)^2=(p-q)^2+4pq$$

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$$f(x)=(x^2+sx+p)(x^2-sx+q)$$

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$$(s^2-10)^2=\frac{144}{s^2}+4(-2)$$

$$s^2(s^2-10)^2=144-8s^2$$

$$s^2(s^2-10)^2+8s^2-144=0$$

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$$s^2(s^2-10)^2+s^2(8)-144=0$$

$$s^2\left[s^4-20s^2+100+8\right]-144=0$$

$$s^2(s^4-20s^2+108)-144=0$$

$$s^6-20s^4+108s^2-144=0$$

Let $y=s^2$

$$y^3-20y^2+108y-144=0$$

$y=0$

$y=1$

$y=2$

Try $y=2$

$$8 - 20(4) + 108(2) - 144 =$$

$$8 - 80 + 216 - 144 =$$

$$224 - 224 = 0$$

$y = 2$ is a root.

$$(y-2)(y^2 + by + c) = y^3 - 20y^2 + 108y - 144$$

(y^3) $a = 1$

(y^0) $c = 72$

(y^2): $-2y^2 + by^2 = -20y^2$

$$-2 + b = -20$$

$$b = -18$$

(y^1): $-2by + cy = 108y$

$$-2(-18) + 72 = 108$$

$$36 + 72 = 108$$

$$LHS = RHS$$

$$(s^2 - 2)(s^4 - 18s^2 + 72) = 0$$

72

9 8

12 6

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$$(s^2 - 2)(s^2 - 12)(s^2 - 6) = 0$$

$s^2 = 2$ OR $s^2 = 12$ OR $s^2 = 6$

use $s^2 = 2$ $\quad s = \pm\sqrt{2}$

Take $s = \sqrt{2}$

$$(x^2 + \sqrt{2}x + p)(x^2 - \sqrt{2}x + q) = f(x) = 0$$

$$x^4 - 10x^2 + 12x - 2 = 0$$

$p + q - s^2 = -10$ — (1)

$(q - p)s = 12$ — (2)

$pq = -2$ (3)

$p+q = 2-10$ from eqt ①

$p+q = -8$

$+ \quad -p+q = \frac{12}{\sqrt{2}}$ from eqt ②

$2q = \frac{12}{\sqrt{2}} - 8$

$2q = \frac{12\sqrt{2}}{2} - 8$

$q = 3\sqrt{2} - 4$

$p+q = -8$

$p + 3\sqrt{2} - 4 = -8$

$p = -4 - 3\sqrt{2}$

$q = -4 + 3\sqrt{2}$

$S = \sqrt{2}$

4 (i) Find integers m and n such that $\sqrt{3+2\sqrt{2}} = m + n\sqrt{2}$. (i)

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$(x^2 + \sqrt{2}x + -4 - 3\sqrt{2})(x^2 - \sqrt{2}x - 4 + 3\sqrt{2}) = 0$

$x^2 + \sqrt{2}x - 4 - 3\sqrt{2} = 0$

$\frac{-\sqrt{2} \pm \sqrt{\sqrt{2}^2 - 4(1)(-4-3\sqrt{2})}}{2} = x$

OR

$$\frac{-\sqrt{2} \pm \sqrt{2+4(4+3\sqrt{2})}}{2} = x$$

$$\frac{-\sqrt{2} \pm \sqrt{18+12\sqrt{2}}}{2} = x$$

$$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$$

$$\frac{-\sqrt{2} \pm \sqrt{6}\sqrt{3+2\sqrt{2}}}{2} = x$$

$$\frac{-\sqrt{2} \pm \sqrt{6}\,(1+\sqrt{2})}{2}$$

$$x = \frac{-\sqrt{2} + \sqrt{6}\,(1+\sqrt{2})}{2} \quad \text{OR} \quad \frac{-\sqrt{2}-\sqrt{6}-2\sqrt{3}}{2}$$

$$x = \frac{-\sqrt{2}+\sqrt{6}+\sqrt{12}}{2}$$

$$x = \frac{-\sqrt{2}+\sqrt{6}+2\sqrt{3}}{2}$$

$$(x^2-\sqrt{2}x-4+3\sqrt{2}) = 0$$

$$\frac{\sqrt{2} \pm \sqrt{2-4(1)(-4+3\sqrt{2})}}{2} = x$$

$$\frac{\sqrt{2} \pm \sqrt{2+16-12\sqrt{2}}}{2} = x$$

$$x = \frac{\sqrt{2} \pm \sqrt{18-12\sqrt{2}}}{2}$$

$$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$$

$1+2\sqrt{2}+2$

$$x = \frac{\sqrt{2} \pm \sqrt{6}\sqrt{3-2\sqrt{2}}}{2}$$

$$\sqrt{3-2\sqrt{2}} = 1-\sqrt{2}$$

$1-2\sqrt{2}+2$

$$x = \frac{\sqrt{2} \pm \sqrt{6}(1-\sqrt{2})}{2}$$

$$x = \frac{\sqrt{2} \pm (\sqrt{6}-\sqrt{12})}{2}$$

$$x = \frac{\sqrt{2}+\sqrt{6}-2\sqrt{3}}{2} \qquad x = \frac{\sqrt{2}-\sqrt{6}+2\sqrt{3}}{2}$$

$$x = \frac{-\sqrt{2}+\sqrt{6}+2\sqrt{3}}{2} \quad \text{OR} \quad \frac{-\sqrt{2}-\sqrt{6}-2\sqrt{3}}{2}$$