Further Mathematics B (MEI)

Proof by Mathematical Induction (MEI)
Andrew Chan

Please note that you may see slight differences between this paper and the original.

Candidates answer on the Question paper.

OCR supplied materials:
Additional resources may be supplied with this paper.

Other materials required:
- Pencil
- Ruler (cm/mm)

Duration: Not set

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the boxes above. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions, unless your teacher tells you otherwise.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Where space is provided below the question, please write your answer there.
- You may use additional paper, or a specific Answer sheet if one is provided, but you must clearly show your candidate number, centre number and question number(s).

INFORMATION FOR CANDIDATES

- The quality of written communication is assessed in questions marked with either a pencil or an asterisk. In History and Geography a *Quality of extended response* question is marked with an asterisk, while a pencil is used for questions in which *Spelling, punctuation and grammar and the use of specialist terminology* is assessed.
- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- The total number of marks for this paper is **128**.
- The total number of marks may take into account some 'either/or' question choices.

1 Prove by induction that $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$.

[8]

2

(i) Use standard series formulae to show that

$$\sum_{r=1}^{n}[r(r-1)-1] = \frac{1}{3}n(n+2)(n-2). \qquad (*)$$

[5]

(ii) Prove (*) by mathematical induction.

[7]

3 Prove by induction that $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.

[7]

4 A sequence is defined by $u_1 = 3$ and $u_{n+1} = 3u_n - 5$. Prove by induction that $u_n = \frac{3^{n-1}+5}{2}$.

[6]

5 A sequence is defined by $u_1 = 8$ and $u_{n+1} = 3u_n + 2n + 5$. Prove by induction that $u_n = 4(3^n) - n - 3$.

[6]

6

You are given that matrix $\mathbf{M} = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$.

(a) Prove that, for all positive integers *n*, $\mathbf{M}^n = \begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}$. **[6]**

(b) Determine the equation of the line of invariant points of the transformation represented by the matrix **M**. **[3]**

It is claimed that the answer to part **(b)** is also a line of invariant points of the transformation represented by the matrix $\mathbf{M}^n$, for any positive integer n.

(c) Explain *geometrically* why this claim is true. **[2]**

(d) Verify *algebraically* that this claim is true. **[3]**

7 **(a)** It is conjectured that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n-1}{n!}=a-\frac{b}{n!},$$

where a and b are constants, and n is an integer such that $n \geq 2$.

By considering particular cases, show that if the conjecture is correct then $a = b = 1$. **[2]**

(b) Use induction to prove that

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+...+\frac{n-1}{n!}=1-\frac{1}{n!} \text{ for } n \geq 2.$$ **[7]**

8 A sequence u_n is defined by $u_{n+1} = 2u_n + 3$ and $u_1 = 1$.

Prove by induction that $u_n = 4 \times 2^{n-1} - 3$ for all positive integers n. **[5]**

9

Prove by induction that $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \left(\frac{1}{2}\right)^n (2+n)$. **[7]**

10 The smallest of three consecutive positive integers is n. Find the difference between the squares of the smallest and largest of these three integers, and hence prove that this difference is four times the middle one of these three integers. **[4]**

11

Matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & c \\ -c & 1 \end{pmatrix}$.

(a) Given that det **M** = 0, find the possible values of c. **[2]**

(b) Prove by induction that $\mathrm{M}^n = 2^{n-1}\mathrm{M}$, for all positive integers n. **[7]**

12

Prove by induction that the sum of the first **n** cube numbers is $\frac{1}{4}n^2(n+1)^2$. **[5]**

13 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n. **[7]**

14

Prove by induction that, for all positive integers n, $\sum_{r=1}^{n} \frac{1}{3^r} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$. **[6]**

15

It is conjectured that $\sum_{r=1}^{n}(r+1)2^r = (a+bn)2^n$, where a and b are constants.

(a) If the conjecture is true, verify that a = 0 and b = 2. **[3]**

(b) Prove the conjecture. **[5]**

16 De Moivre's theorem states that, for all positive integers n, $(\cos \theta + \mathrm{i} \sin \theta)^n = \cos n\theta + \mathrm{i} \sin n\theta + \mathrm{i} \sin n\theta$.

(a) Prove this result by induction. **[5]**

(b) By considering $\frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^n}$, extend the proof to negative integers. **[4]**

17

Prove by induction that $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & 2^n - 1 \\ 0 & 2^n \end{pmatrix}$ for all positive integers n. **[6]**

END OF QUESTION PAPER

Question			Answer/Indicative content	Marks	Part marks and guidance	
1			When $n = 1,\ (-1)^0 \frac{1\times 2}{2} = 1$	B1		
			and $1^2 = 1$, so true for $n = 1$ Assume true for $n = k$	B1		
			$\Rightarrow 1^2 - 2^2 + 3^2 - \ldots\ldots + (-1)^{k-1}k^2 = (-1)^{k-1}\frac{k(k+1)}{2}$	E1	Assuming true result for some n.	Condone series shown incomplete
			$\Rightarrow 1^2 - 2^2 + 3^2 - \ldots\ldots + (-1)^{k-1}k^2 + (-1)^{k+1-1}(k+1)^2$ $= (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^{k+1-1}(k+1)^2$	M1*	Adding (k + 1)th term to both sides.	
			$= (-1)^k\left[\frac{-k(k+1)}{2} + (k+1)^2\right]$	M1 Dep*	Attempt to factorise (at least one valid factor)	
			$= (-1)^k(k+1)\left(\frac{-k}{2} + k + 1\right)$	A1	Correct factorisation Accept $(-1)^{k\pm m}$ provided expression correct.	
			$= (-1)^k(k+1)\left(\frac{k+2}{2}\right)$	A1	Valid simplification with $(-1)^k$	
			$= (-1)^{[n-1]}\frac{n(n+1)}{2},\ n = k+1$	E1	Or target seen	
			Therefore if true for $n = k$ it is also true for $n = k + 1$		Dependent on A1 and previous E1	
			Since it is true for $n = 1$, it is true for all positive integers, n.	E1	Dependent on B1 and previous E1 **Examiner's Comments** Most candidates knew what to do and were meticulous in presenting their argument. The factor $(-1)^k$was successfully dealt with by many although it caused a problem for some. The precise wording needed to round off the argument was in the main well expressed.	

Question			Answer/Indicative content	Marks	Part marks and guidance
			Total	8	

Question			Answer/Indicative content	Marks	Part marks and guidance	
2		i	$\sum_{r=1}^{n}[r(r-1)-1]=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r-n$	M1	Split into separate sums	
		i	$=\frac{1}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)-n$	M1	Use of at least one standard result (ignore 3rd term)	
		i		A1	Correct	
		i	$=\frac{1}{6}n[(n+1)(2n+1)-3(n+1)-6]$	M1	Attempt to factorise. If more than two errors, M0	
		i	$=\frac{1}{6}n[2n^2-8]$			
		i	$=\frac{1}{3}n[n^2-4]$	A1	Correct with factor $\frac{1}{3}n$ oe	
		i	$=\frac{1}{3}n(n+2)(n-2)$		Answer given **Examiner's Comments** This was usually answered well. The most common error was to write $\sum_{r=1}^{n}1=1$ and this led to difficulty with earning the next mark, especially when compounded by trying to work back from the given result. This question was also subject to some careless notation; too few sigmas and missing brackets.	
		ii	When n = 1, $\sum_{r=1}^{n}[r(r-1)-1]=(1\times 0)-1=-1$ and $\frac{1}{3}n(n+2)(n-2)=\frac{1}{3}\times 1\times 3\times -1=-1$			
		ii	So true for n = 1	B1		

Question			Answer/Indicative content	Marks	Part marks and guidance	
		ii	Assume true for $n = k$ $\sum_{r=1}^{k}\left[r(r-1)-1\right] = \frac{1}{3}k(k+2)(k-2)$	E1	Or "if true for n = k, then…"	
		ii	$\Rightarrow \sum_{r=1}^{k+1}\left[r(r-1)-1\right] = \frac{1}{3}k(k+2)(k-2)+(k+1)k-1$ $= \frac{1}{3}k^3 + k^2 - \frac{4}{3}k + k - 1$ $= \frac{1}{3}\left(k^3 + 3k^2 - k - 3\right)$	M1*	Add (k + 1)th term to both sides	
		ii	$= \frac{1}{3}(k+1)\left(k^2 + 2k - 3\right)$	M1dep*	Attempt to factorise a cubic with 4 terms	
		ii	$= \frac{1}{3}(k+1)(k+3)(k-1)$	A1		
		ii	$= \frac{1}{3}(k+1)((k+1)+2)((k+1)-2)$		Or = $\frac{1}{3}n(n+2)(n-2)$ where $n = k + 1$; or target seen	
		ii	But this is the given result with $n = k + 1$ replacing $n = k$. Therefore if the result is true for $n = k$, it is also true for $n = k + 1$.	E1	Depends on A1 and first E1	

Question			Answer/Indicative content	Marks	Part marks and guidance	
		ii	Since it is true for n = 1, it is true for all positive integers, n.	E1	Depends on B1 and second E1 Examiner's Comments Most candidates knew how to earn the first three marks, but again the written work was frequently scruffy with missing sigmas in particular. It is nonsense to write the sum of a series as equal to its last term. In some scripts, the added term in the series was the k th, not the (k+1) th. The following algebra proved too much for quite a few candidates, again not helped by missing brackets. It was just about possible to believe that the correct four term cubic could be instantly factorised, and some benefit of the doubt was given here. Inevitably, marks were lost in the details of the induction argument. Initially, "assume n = k " does not state what is being assumed. "True for n = 1 and n=k" is not so, when the latter is conditional. The language must be precise, and many candidates displayed only half remembered sentences, indicating that they did not fully understand the induction argument.	
			Total	12		

Question			Answer/Indicative content	Marks	Part marks and guidance	
3			When $n = 1$, $\frac{1}{1\times 3}=\frac{1}{3}$	B1	Condone eg "$\frac{1}{3}=\frac{1}{3}$"	
			and $\frac{n}{2n+1}=\frac{1}{3}$, so true for $n = 1$			
			Assume true for $n = k$	E1	Assuming true for k, (some work to follow) If in doubt look for unambiguous "if…then" at next E1 Statement of assumed result not essential but further work should be seen	
			Sum of $k + 1$ terms		NB "last term = sum of terms" seen anywhere earns final E0	
			$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$	M1	Adding correct (k+1)th term to sum for k terms	
			$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$	M1	Combining their fractions	
			$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$			
			$=\frac{(k+1)(2k+1)}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$	A1	Complete accurate work	
			which is $\frac{n}{2n+1}$ with $n = k + 1$		May be shown earlier	
			Therefore if true for $n = k$ it is also true for $n = k + 1$.	E1	Dependent on A1 and previous E1.	

Question			Answer/Indicative content	Marks	Part marks and guidance	
			Since it is true for n = 1, it is true for all positive integers, n	E1	Dependent on B1 and previous E1 E0 if "last term" = "sum of terms" seen above **Examiner's Comments** It was good to see many extremely well argued proofs. Very many candidates showed that they appreciated the need for clarity and logical statements, but the three explanation marks were not always earned. Proof by induction has a standard format: prove for n = 1; assume the conjecture is true for n = k and hence show it is true for n = k + 1; state "if it is truefor n = k then it is true for n = k + 1; since it is true for n = 1…" etc. The mark schemes of past years set out the argument clearly but there are still those who believe that "n = k" is sufficient to tell the reader that a result is being assumed to be true, and that "true for" n = 1,n = k and n = k + 1 is adequate to replace "if…then…since…". This particular series should not have been difficult to deal with algebraically. Candidates could help themselves by choosing the simplest denominator when combining fractions. When this was not done and a cubic obtained by expanding the numerator, it was not always convincingly re-factorised. Somecandidates made the	

Question			Answer/Indicative content	Marks	Part marks and guidance	
					mistake of adding $\frac{1}{2k\;2k\quad 2}$ as the k + 1 th term. Some lost a mark because, in their work, the last term of the series was equated to the sum of the terms. Is this laziness, or a lack of understandingof the notation?	
			Total	7		

Question			Answer/Indicative content	Marks	Part marks and guidance	
4			$u_1 = 3$ and $\frac{3^{1-1}+5}{2} = 3$, so true for $n = 1$	B1	Must show working on given result with $n = 1$	
			Assume true for $n = k$ $\Rightarrow u_k = \frac{3^{k-1}+5}{2}$	E1	Assuming true for k Allow "Let $n = k$ and (result)" "If $n = k$ and (result)" Do not allow "$n = k$" or "Let $n = k$", without the result quoted, followed by working	
			$\Rightarrow u_{k+1} = 3\left(\frac{3^{k-1}+5}{2}\right) - 5$ $= \frac{3^k+15}{2} - 5$ $= \frac{3^k+15-10}{2}$	M1	u_{k+1} with substitution of result for uz and some working to follow	
			$= \frac{3^k+5}{2}$	A1	Correctly obtained	
			$= \frac{3^{n-1}+5}{2}$ when $n = k + 1$		Or target seen	
			Therefore **if** true for $n = k$ it is **also true** for $n = k + 1$.		Both points explicit	
				E1	Dependent on A1 and previous E1	

Question			Answer/Indicative content	Marks	Part marks and guidance	
			Since it is true for n = 1, it is true for all positive integers, n.	E1	Dependent on B1 and previous E1 **Examiner's Comments** Yet again the number of students who fail to follow the recipe for Induction is surprising. The candidates split into four camps. 1) Those who understood the idea of induction and did this perfectly. Good work. This question was well answered by the majority of candidates who had learned the required argument and expressed it well. 2) Others who also had the precise wording for the start and the finish, and wrote this down even though they did not have a middle section that supported their final assertions. 3) The third group, who got as far showing the n=k+1 term is in the desire form and then failing to be precise about the final wording. 4) Lastly those who did not seem to understand induction at all. There werenot many of these. The first four marks were usually obtained, with some latitude allowed for different ways of expressing the assumption of “true for n =	

Question			Answer/Indicative content	Marks	Part marks and guidance	
					k", (not the same thing as "assume $n = k$ is true"). Apart from some candidates who believed that they were dealing with a series, the central passage of algebra was successfully done. However the number of students who lost the final two explanation marks is extremely disappointing since this point is raised every year. The wording that is unarguably acceptable has been set out in the mark schemes for many years. If it is true when n=k it is also true when n=k+1. Since it is true for n=1, it is true for all positive integers n.".	
			Total	6		

Question			Answer/Indicative content	Marks	Part marks and guidance	
5			When $n = 1$, $u_n = 4(3)^1 - 1 - 3 = 8$, (so true for $n = 1$)	B1	Showing use of $u_n = 4(3)^n - n - 3$	
			Assume $u_k = 4(3)^k - k - 3$	E1	Assuming true for $n = k$ Allow "let $n = k$ and (result)" or "If $n = k$ and (result)" Do not allow "$n = k$" or "let $n = k$" without the result quoted	
			$\Rightarrow u_{k+1} = 3u_k + 2k + 5 = 3(4(3)^k - k - 3) + 2k + 5$	M1	u_{k+1} using u_k and attempting to simplify	
			$= 4(3)^{k+1} - (k + 1) - 3$	A1	Correct simplification or identification with a 'target' expression using $n = k + 1$ The "target" shows this	
			But this is the given result with $k + 1$ replacing k. Therefore **if** it is true for $n = k$ **then** it is also true for $n = k + 1$.	E1	Dependent on A1 and previous E1	
			Since it is true for $n = 1$, it is true for all positive integers.	E1	Dependent on B1 and previous E1 **Examiner's Comments** An encouraging number of complete logical arguments were seen, but many candidates lost the final two marks for inadequate explanation. It is incorrect to claim the result is "true for $n = k + 1$" before both pointing out the structure and conditioning on the assumption. Using abbreviations such as "$n = k + 1$ is true" is also nonsensical and insufficient.	
			Total	6		

Question			Answer/Indicative content	Marks	Part marks and guidance	
6		a	$\mathbf{M}^1=\begin{pmatrix}1-4 & 8\\-2 & 1+4\end{pmatrix}=\begin{pmatrix}-3 & 8\\-2 & 5\end{pmatrix}$ Assume true for *n* = *k*: $\mathbf{M}^k=\begin{pmatrix}1-4k & 8k\\-2k & 1+4k\end{pmatrix}$ $\mathbf{M}^{k+1}=\mathbf{M}^k\times\mathbf{M}=\begin{pmatrix}1-4k & 8k\\-2k & 1+4k\end{pmatrix}\begin{pmatrix}-3 & 8\\-2 & 5\end{pmatrix}$ $=\begin{pmatrix}-3-4k & 8+8k\\-2-2k & 5+4k\end{pmatrix}$ $=\begin{pmatrix}1-4(k+1) & 8(k+1)\\-2(k+1) & 1+4(k+1)\end{pmatrix}$ which is the formula with *n* = *k* + 1. Therefore if true for *n*, true for *n* + 1. True for 1 ⇒ true for 2, 3,… and all positive integers	B1(AO1.1) E1(AO2.1) M1(AO1.1) A1(AO2.1) A1(AO2.4) E1(AO2.2a) [6]	OR $\mathbf{M}^{k+1}=\mathbf{M}\times\mathbf{M}^k=\begin{pmatrix}-3 & 8\\-2 & 5\end{pmatrix}\begin{pmatrix}1-4k & 8k\\-2k & 1+4k\end{pmatrix}$ Not required to check both ways round. Convincingly expressed in terms of *k* + 1 Completion of proof by induction. Dependent on **B1** and previous **E1**	
		b	$\begin{pmatrix}-3 & 8\\-2 & 5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ $\Rightarrow\ -3x+8y=x,\ \ -2x+5y=y$ $\Rightarrow\ y=\frac{1}{2}x$	M1(AO1.1a) M1(AO1.1) A1(AO2.2a) [3]	Using **Mx** = **x** Obtaining an equation Any correct form; must state both equations lead to this answer.	

Question			Answer/Indicative content	Marks	Part marks and guidance	
		c	M^n represents the transformation repeated n times. Each repeat leaves points on $y = \frac{1}{2}x$ unchanged.	E1(AO2.4) E1(AO2.1) [2]		
		d	$\begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix}$ $= \begin{pmatrix} x-4nx+8n(\frac{1}{2}x) \\ -2nx+\frac{1}{2}x+4n(\frac{1}{2}x) \end{pmatrix}$ $= \begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix}$ so same interval line for all n.	M1(AO1.1) A1(AO1.1) E1(AO2.3) [3]	Multiplication with some progress cao	
			Total	14		

Question			Answer/Indicative content	Marks	Part marks and guidance	
7		a	E.g. n = 2 gives $\frac{1}{2}=a-\frac{b}{2}$ and n = 3 gives $\frac{5}{6}=a-\frac{b}{6}$ Verify (or solve) $a = b = 1$	M1(AO3.1a) A1(AO2.2a) [2]	Two values of $n \geq 2$ to give two simultaneous equations	
		b	Result holds for $n = 2\left(\frac{1}{2}=\frac{1}{2}\right)$ Assume $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k-1}{k!}=1-\frac{1}{k!}$ $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k-1}{k!}+\frac{k}{(k+1)!}$ $=1-\frac{1}{k!}+\frac{k}{(k+1)!}$ RHS is $1-\frac{1}{k!}+\frac{k}{(k+1)!}=1-\left(\frac{(k+1)-k}{(k+1)!}\right)$ which is $1-\left(\frac{1}{(k+1)!}\right)$ The result is true for n = 2. If true for $n = k$ it is also true for k + 1 hence true for n = 2, 3, 4, …	B1(AO1.1) E1(AO2.1) M1A1(AO2.1 AO1.1) M1(AO1.1) A1(AO2.2a) E1(AO2.5) [7]	Assume true for $n = k$ Add correct next term to both sides 1 minus fraction with correct denominator on RHS Showing that this is the given result with k + 1 replacing k Complete argument	
			Total	9		

Question			Answer/Indicative content	Marks	Part marks and guidance		
8			When $n = 1$, $u_1 = 4\times 2^0 - 3 = 1$, as required	B1(AO 1.1)			
			Assume true for $n = k$, so $u_K = 4\times2^{k-1} - 3$	E1(AO 2.1)			
			Then $u_{k+1} = 2(4\times 2^{k-1} - 3) + 3 = 4\times2^k - 6 + 3$	M1(AO 2.1)	Use of relation, with some simplificatio n		
			$= 4\times2k - 3$, which is the formula for $n = k + 1$	A1(AO 2.4)	Correct result reached and identified		
			So if true for $n = k$, also true for $n = k + 1$; therefore true for all positive integers n	E1(AO 2.2a) [5]	Dependent on the previous A1 and the initial B1 marks both being earned		
			Total	5			

Question				Answer/Indicative content	Marks	Part marks and guidance	
9				When n = 1, LHS $= \frac{1}{2}$ and $\text{RHS} = 2-\left(\frac{1}{2}\right)^1(2+1) = 2-\frac{3}{2} = \frac{1}{2}$ so true for n = 1	B1		
				Assume true for $n = k$ that is $\sum_{r=1}^{k}\frac{r}{2^r} = 2-\left(\frac{1}{2}\right)^k(2+k)$	E1	Assume true for $n = k$	
				$\sum_{r=1}^{k+1}\frac{r}{2^r} = 2-\left(\frac{1}{2}\right)^k(2+k)+\frac{k+1}{2^{k+1}}$	M1*	Add correct term to RHS followed by working	
				$= 2-\left(\frac{1}{2}\right)^k\left(2+k-\left(\frac{k+1}{2}\right)\right)$ $= 2-\left(\frac{1}{2}\right)^k\left(\frac{4+2k-k-1}{2}\right)$	M1dep*	Attempt to factorise with a factor of $\left(\frac{1}{2}\right)^k$ or $\left(\frac{1}{2}\right)^{k+1}$	
				$= 2-\left(\frac{1}{2}\right)^{k+1}(2+(k+1))$ or $2-\left(\frac{1}{2}\right)^{k+1}(k+3)$ **wih target seen**	A1	cao with correct working	
				Therefore **if** it is true for $n = k$ **then** it is true for $n = k + 1$.	E1	Dependent on A1 and first E1	
				It is true for n = 1 therefore it is true for n = 1,2,3,… (and so is true for all positive integers).	E1	Dependent on B1 and second E1	
					[7]	**Examiner's Comments** A substantial minority demonstrated an inability to cope with indices. This meant that, for many, this question was not about demonstrating the concept of proof by induction, but about index manipulation. The algebra proved too difficult for most. In particular, a minus sign outside the bracket was	

Question			Answer/Indicative content	Marks	Part marks and guidance	
					easily the most common error here. Bad handwriting meant that indices often got confused, e.g. the index numbers were written in line with everything else and then became coefficients instead of indices. In the denominator for example, 2^{k+1} often became $2^k + 1$ or $2(k + 1)$ or even $2k + 1$ in later work. Some multiplied out and created very complicated expressions, often carrying through terms such as 1^{-2k} for several lines seemingly unaware that it could be simplified. Rough working was interspersed between lines of their deductive steps without making it clear what they were doing. We often saw a whole line multiplied by 2^{k+1} and later if we were lucky, divided by 2^{k+1} with no explanation given. The most successful were those who started by creating two fractions with clear denominators of 2^k and 2^{k+1}, and developed the work from there. Many candidates write the last statements down even if they have not done previous work. They need to be aware that this wastes their time because of the way this question is marked, where previous steps need to be correct. Apart from these points, it was good to see that a great many candidates had	

Question			Answer/Indicative content	Marks	Part marks and guidance	
					learned the appropriate introductions to the proof, and where they were successful in the algebraic argument, followed up with the crucial inductive argument, earning full marks. There were still a few candidates who insisted that their result showed that "$n = k + 1$ is true".	
			Total	7		

Question			Answer/Indicative content	Marks	Part marks and guidance		
10			n $n + 1$ $n + 2$ soi	B1	may be earned later		
			$(n + 2)^2 – n^2$ soi	M1	allow ft for next three marks for other general consecutive integers eg $n – 1$ n $n + 1$	allow $n^2 – (n + 2)^2$ for M1 then A0 for negative answer; may still earn last B1	
			$4n + 4$ obtained with at least one interim step shown	A1	for other integers in terms of n (eg $2n$, $2n + 1$, $2n + 2$ or $2n + 1$, $2n + 3$, $2n + 5$) allow ft for this M1 only		
			$4(n + 1)$ or $\frac{4n+4}{4} = n + 1$	B1		B0 for $n + 1 \times 4$	
				[4]	may be obtained independently		
					Examiner's Comments The majority of candidates that attempted this standard proof question gained full marks, showing the needed interim step(s) to obtain the corresponding accuracy marks. A minority chose wrong expressions for the three integers (e.g. n, $2n$, $3n$). Unfortunately candidates missing the middle term of $4n$ when squaring the $(n + 2)$ term was seen quite often. Some candidates		

Question			Answer/Indicative content	Marks	Part marks and guidance	
					considered the first term squared minus the last term squared and then conveniently ignored the negative signs. A handful of candidates attempted an entirely numerical approach.	
			Total	4		

Question			Answer/Indicative content	Marks	Part marks and guidance	
11		a	det **M** = $c^2 + 1$	B1 (AO 1.1a)		
			so c = i or –i	B1 (AO 1.1)		
				[2]		
		b	When $n = 1$, $\mathbf{M}^1 = 2^0\mathbf{M} = \mathbf{M}$, as required	B1 (AO 2.1)	Check of initial case	
			Assume true for $n = k$, so $\mathbf{M}^k = 2^{k-1}\mathbf{M}$	M1 (AO 2.1)	Statement of induction hypothesis	
				M1 (AO 2.1)	Or using c = –i	May use matrix with c and c^2 = –1 to cover both cases for M1A1B1
			If c = i, then $\mathbf{M}^{k+1} = 2^{k-1}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}$	A1* (AO 2.2a)		
			$= 2^{k-1}\begin{pmatrix}2 & 2\mathrm{i}\\ -2\mathrm{i} & 2\end{pmatrix} = 2^k\begin{pmatrix}1 & \mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix} = 2^k\mathbf{M}$			
			If c = –i, then $\mathbf{M}^{k+1} = 2^{k-1}\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}$		Or using c = i	
			$= 2^{k-1}\begin{pmatrix}2 & -2\mathrm{i}\\ 2\mathrm{i} & 2\end{pmatrix} = 2^k\begin{pmatrix}1 & -\mathrm{i}\\ \mathrm{i} & 1\end{pmatrix} = 2^k\mathbf{M}$	B1 (AO 2.1)		
			so if true for $n = k$ then also true for $n = k + 1$	B1dep* (AO 2.2a)		
			therefore true for all positive integers n	B1dep* (AO 2.5)		
				[7]		
			Total	9		

Question			Answer/Indicative content	Marks	Part marks and guidance		
12			When $n = 1$, $1^3 = \frac{1}{4} \times 1^2 \times (1+1)^2$ so true for $n = 1$ Assume true for $n = k$: $\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k+1)^2$ Then $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$ $= \frac{1}{4}(k+1)^2(k^2+4k+4)$ $= \frac{1}{4}(k+1)^2(k+2)^2$ which is the formula for $n = k + 1$ So if true for $n = k$, true also for $n = k + 1$ therefore true for all $n \geq 1$	B1 (AO2.1) M1 (AO2.1) M1 (AO2.1) A1 (AO2.2a) E1 (AO2.4) [5]	Dependent on all previous marks earned		
			Total	5			

Question			Answer/Indicative content	Marks	Part marks and guidance	
13			When $n = 1$, $5^1 + 2\times11^1 = 27$ div by 3 Assume $u_k = 5^k + 2\times11^k$ is div by 3 $u_{k+1} = 5^{k+1} + 2\times11^{k+1}$ $= 5(u_k - 2\times11^k) + 22\times11^k$ $= 5u_k + 12\times11^k$ or $u_{k+1} = 5^{k+1} + 11(u_k - 5^k)$ $= 11u_k - 6\times5^k$ or $u_{k+1} + u_k = 5^{k+1} + 2\times11^{k+1} + 5^k + 2\times11^k$ $= 6\times5^k + 24\times11^k$ As u_k div by 3, u_{k+1} div by 3 So if true for $n = k$, true for $n = k+1$. As true for $n = 1$, true for all positive integers n	B1*(AOs 2.1) M1(AOs2.1) M1 M1 A1(AOs1.1b) M1 A1 M1(AOs1.1b) A1 A1*(AOs 2.2a) A1dep(AOs2.4) [7]	or $5^k + 2\times11^k = 3m$ substituting for 5^k or $15m + 12\times11^k$ substituting for 11^k or $33m - 6\times5^k$ adding u_k to u_{k+1} dep* marks **Examiner's Comments** The inductive step for divisibility proofs can be quite tricky and we found some interesting variations for negotiating this. The structure of an inductive proof was generally well understood, and most candidates who overcame the inductive step gained full marks. We expect a careful summary statement of the inductive principle at the end to achieve the final 'A' mark, along the lines of 'as it is true for $n = 1$, and if true for $n = k$ then true for $n = k+1$, it is therefore true for all n'. **AfL**	$5.5^k + 11(3m - 5^k)$

Question			Answer/Indicative content	Marks	Part marks and guidance	
					It is worth emphasising the technique of substituting for or extracting multiples of $3m - 5\times11^k$ for 5^k (or its equivalent) in induction proofs of divisibility like this.	
			Total	7		

<table>
<tr><th colspan="3">Question</th><th>Answer/Indicative content</th><th>Marks</th><th colspan="2">Part marks and guidance</th></tr>
<tr><td>14</td><td></td><td></td><td>When n = 1, $\frac{1}{3}=\frac{1}{2}\left(1-\frac{1}{3}\right)$
so true for n = 1

Assume true for n = k so
$\sum_{r=1}^{k}\frac{1}{3^r}=\frac{1}{2}\left(1-\frac{1}{3^k}\right)$
s $\sum_{r=1}^{k+1}\frac{1}{3^r}=\frac{1}{2}\left(1-\frac{1}{3^k}\right)+\frac{1}{3^{k+1}}$
o

$=\frac{1}{2}\left(1-\frac{3}{3^{k+1}}+\frac{2}{3^{k+1}}\right)$
$=\frac{1}{2}\left(1-\frac{1}{3^{k+1}}\right)$[so true for n = k + 1]

True for n = k ⇒ true for n = k + 1⇒ true for all n</td><td>B1(AO 2.1)

M1(AO 2.1)

M1(AO 2.1)

A1(AO 1.1)

A1*(AO 2.2a)
E1dep (AO 2.4)

[6]</td><td>condone notation errors

addi $\frac{1}{3^{k+1}}$ to
ng
$\frac{1}{2}\left(1-\frac{1}{3^k}\right)$

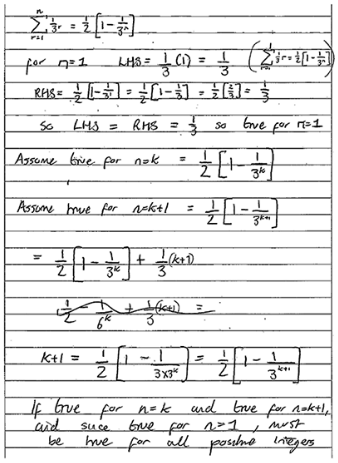
combining fractions

dep A1*

<u>Examiner's Comments</u>
The method of mathematical induction was well rehearsed by the majority of candidates. However, the inductive step proved to be tricky for many, who were unable to successfully combine the fracti $\frac{1}{3^k}$ an $\frac{1}{3^{k+1}}$. This
ons d proved
rather costly, as this step was required to achieve the concluding A1 mark.

AfL

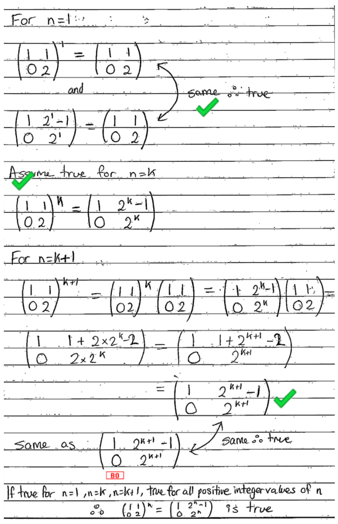
The use of precise language to support the algebraic manipulation is a key part of any proof by induction. It is also important that candidates</td><td></td></tr>
</table>

Question			Answer/Indicative content	Marks	Part marks and guidance	
					understand that the truth for n = k is an assumption, not a given. **Exemplar 3**  $\sum_{r=1}^{n}\frac{1}{3^r}=\frac{1}{2}\left[1-\frac{1}{3^n}\right]$ for $n=1$ LHS $=\frac{1}{3}(1)=\frac{1}{3}$ $\left(\sum_{r=1}^{1}\frac{1}{3^r}=\frac{1}{2}\left[1-\frac{1}{3^1}\right]\right)$ RHS $=\frac{1}{2}\left[1-\frac{1}{3^1}\right]=\frac{1}{2}\left[1-\frac{1}{3}\right]=\frac{1}{2}\left[\frac{2}{3}\right]=\frac{1}{3}$ so LHS = RHS = $\frac{1}{3}$ so true for $n=1$ Assume true for $n=k$ $=\frac{1}{2}\left[1-\frac{1}{3^k}\right]$ Assume true for $n=k+1$ $=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$ $=\frac{1}{2}\left[1-\frac{1}{3^k}\right]+\frac{1}{3}(k+1)$ $k+1=\frac{1}{2}\left[1-\frac{1}{3\times3^k}\right]=\frac{1}{2}\left[1-\frac{1}{3^{k+1}}\right]$ If true for $n=k$ and true for $n=k+1$, and since true for $n=1$, must be true for all positive integers This candidate has assumed the truth of the statement for *n* = *k* + 1, instead of attempting to prove its truth assuming the truth for *n* = *k*. The inductive step is missing, and the final statement is incorrect.	
			Total	6		

Question			Answer/Indicative content	Marks	Part marks and guidance
15		a	$n = 1$: $4 = 2a + 2b \Rightarrow a + b = 2$ $n = 2$: $16 = 4a + 8b \Rightarrow a + 2b = 4$ so $a = 0$ and $b = 2$	B1 (AO 2.1) B1 (AO 1.1) B1 (AO 1.1) [3]	Substituting any value of n Substituting any second value of n **AG** Working must be clear
		b	When $n = 1$, result is true from part **(a)** Assume true for $n = k$, so $\sum_{r=1}^{k}(r+1)2^r = k \times 2^{k+1}$ $\sum_{r=1}^{k+1}(r+1)2^r = k \times 2^{k+1} + (k+2) \times 2^{k+1}$ Then $(2k + 2) \times 2^{k+1} = (k + 1) \times 2^{(k+1)+1}$, so true for $n = k + 1$ So if true for $n = k$, true for $n = k + 1$; and as true for $n = 1$, true for all positive n	B1 (AO 2.1) M1 (AO 2.1) M1 (AO 2.1) E1 (AO 2.4) E1 (AO 2.2a) [5]	Or direct check if $n = 1$ not used in **(a)** Correct algebra and statement Proof correctly concluded
			Total	**8**	

Question			Answer/Indicative content	Marks	Part marks and guidance	
16		a	$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$, so true for $n = 1$ assume true for $n = k$: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ $(\cos\theta + i\sin\theta)^{k+1} = (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ $= \cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\sin k\theta + \cos k\theta\sin\theta)$ $= \cos(k+1)\theta + i\sin(k+1)\theta$ so if true for $n = k$ then true for $n = k + 1$ but true for $n = 1$ so true for all positive integers n	B1 (AO 2.1) B1 (AO 2.1) M1 (AO 2.1) A1 (AO 2.2a) E1 (AO 2.4) [5]	verifying for $n = 1$ using result for $n = k$ must end with both statements	
		b	$\frac{1}{(\cos\theta + i\sin\theta)^n} = \frac{1}{\cos n\theta + i\sin n\theta}$ $= \frac{\cos n\theta - i\sin n\theta}{(\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta)} = \frac{\cos n\theta - i\sin n\theta}{\cos^2 n\theta + \sin^2 n\theta}$ $= \cos n\theta - i\sin n\theta = \cos(-n\theta) + i\sin(-n\theta)$ so $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$, and result is proved for negative integers	B1 (AO 3.1a) M1 (AO 1.1) A1 (AO 2.1) E1 (AO 2.2a) [4]	realising the denominato r complete correct argument	
			Total	9		

Question			Answer/Indicative content	Marks	Part marks and guidance	
17			When n $\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^1 = \begin{pmatrix}1 & 2^1-1\\0 & 2^1\end{pmatrix}$ = 1, [Assu $n = k$: $\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^k = \begin{pmatrix}1 & 2^k-1\\0 & 2^k\end{pmatrix}$ me] The $\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}^{k+1} = \begin{pmatrix}1 & 2^k-1\\0 & 2^k\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}$ n $= \begin{pmatrix}1 & 1+2^{k+1}-2\\0 & 2^{k+1}\end{pmatrix} = \begin{pmatrix}1 & 2^{k+1}-1\\0 & 2^{k+1}\end{pmatrix}$ so **if** true for $n = k$ then true for $n = k + 1$ [as true for n = 1] therefore true for all n.	B1 (AO2.1) M1 (AO2.2a) M1 (AO1.1) A1* (AO1.1) M1dep* (AO2.2a) A1dep* (AO2.4) [6]	$\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 2^k-1\\0 & 2^k\end{pmatrix}$ or $\begin{pmatrix}1 & 2^k-1+2^k\\0 & 2^{k+1}\end{pmatrix}$ or must clearly 'assume' $n = k$ must have established truth for n = 1 for final mark	'true for $n = k$ and k + 1' is M0 but need not re-state it at the end
					Examiner's Comments The induction step in this question was a relatively straightforward one. However, it is important that candidates show some working to justify this step. Errors and omissions here proved quite costly, as the final two marks were dependent on this. These errors, though rare, were of two types: one was to add M to Mk (where M is the given matrix); another involved manipulating handling indices, such as writing 2^{k-1} instead of $2^k - 1$, or $2 \times 2^k = 4^k$. In order to gain full marks for this question, it was important that candidates used precise language that showed a	

Question			Answer/Indicative content	Marks	Part marks and guidance	
					clear mathematical justification of the induction process. For example, in their concluding statement, if they wrote ‘so, as it is true for n = 1, n = k and n = k + 1, it is true for all n’, they missed out on the final two marks; whereas if they wrote ‘so, as it is true for n = 1, and **if** true for n = k, then true for n = k + 1, it is true for all n’, they got full marks (assuming the induction step was negotiated without errors). Although many candidates are well schooled in the logic of induction, the use of precise language to convey this logic is required, and it is important that candidates understand that the truth for n = k is an *assumption*, not a given. **Exemplar 2**  In this response, the candidate loses a mark in summarising the induction step for the above reason.	

Question			Answer/Indicative content	Marks	Part marks and guidance
			Total	6	