(1)(a) $$\frac{1+\tanh^2 x}{1-\tanh^2 x} = \frac{1+\tanh^2 x}{\operatorname{sech}^2 x} \qquad (1-\tanh^2 x \equiv \operatorname{sech}^2 x)$$

$$= (1+\tanh^2 x)\cosh^2 x \qquad \left(\cosh^2 x \equiv \frac{1}{\operatorname{sech}^2 x}\right)$$

$$= \left(1+\frac{\sinh^2 x}{\cosh^2 x}\right)\cosh^2 x \qquad \left(\tanh^2 x \equiv \frac{\sinh^2 x}{\cosh^2 x}\right)$$

$$= \cosh^2 x + \sinh^2 x$$

$$= \cosh(2x)$$

$$(\cosh(2x) \equiv \cosh^2(x) + \sinh^2(x))$$

(b) $$\frac{1+\tanh^2 x}{1-\tanh^2 x} = \frac{17}{8}$$

$$\Rightarrow \cosh(2x) = \frac{17}{8}$$

$$\Rightarrow \frac{e^{2x}+e^{-2x}}{2} = \frac{17}{8}$$

$$\Rightarrow 4e^{2x} - 17 + 4e^{-2x} = 0$$

$$\Rightarrow 4e^{4x} - 17e^{2x} + 4 = 0$$

$$\Rightarrow e^{2x} = \frac{1}{4} \quad \text{or} \quad e^{2x} = 4$$

$$\Rightarrow x = \ln\left(\tfrac{1}{2}\right) \quad \text{or} \quad x = \ln(2)$$

(2)(a)(i) $|z-2i| = 2$

(ii) $\arg(z+2) = \frac{\pi}{4}$

(b) $\{z : |z-2i| = 2\} \cap \{z : \arg(z+2) = \frac{\pi}{4}\}$

(c) From the geometry of the problem, we have

$$z = 2i + 2e^{i\frac{\pi}{4}} = \sqrt{2} + (2+\sqrt{2})i$$

or

$$z = 2i + 2e^{i\frac{5\pi}{4}} = -\sqrt{2} + (2-\sqrt{2})i$$

(3)(a) $M = QP = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$ //

(b) $M\vec{v} = \vec{v}$

$\Leftrightarrow \begin{cases} -\frac{1}{2}x - \frac{\sqrt{3}}{2}y = x \\ -\frac{\sqrt{3}}{2}x + \frac{1}{2}y = y \end{cases}$

$\Leftrightarrow \begin{cases} -\frac{3}{2}x - \frac{\sqrt{3}}{2}y = 0 & \text{— } (1) \\ -\frac{\sqrt{3}}{2}x - \frac{1}{2}y = 0 & \text{— } (2) \end{cases}$

(1) $= \sqrt{3} \times$ (2) so the equations are equivalent, hence there is an invariant line of $-\frac{3}{2}x - \frac{\sqrt{3}}{2}y = 0$, i.e. $y = -\sqrt{3}x$ //

(4) (a) $\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= n(n+1)\left(\frac{2n+1}{6} + \frac{1}{2}\right)$$

$$= n(n+1)\,\frac{2n+4}{6}$$

$$= \frac{n}{3}(n+1)(n+2)$$

(b) $\log(9) + 2\log(27) + 3\log(81) + \cdots + 11\log(531441)$

$$= \sum_{r=1}^{11} r\log(3^{r+1})$$

$$= \log(3)\sum_{r=1}^{11} r(r+1)$$

$$= \log(3)\,\frac{11}{3}(11+1)(11+2)$$

$$= 572\log(3)$$

(5) Let A, B, C be his sales in 2015.

So his sales in 2016 are $1.02A$, $1.01B$, $0.955C$

$$\text{Thus, } \begin{cases} A+B+C = 300 \\ B = A+20 \\ 1.02A+1.01B+0.955C = 300-10. \end{cases}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1.02 & 1.01 & 0.955 \end{pmatrix}}_{M} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 300 \\ 20 \\ 290 \end{pmatrix}$$

$$\det(M) = 1 \cdot \det\begin{pmatrix} -1 & 1 \\ 1.02 & 1.01 \end{pmatrix} + 0.955 \cdot \det\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= -1.01 - 1.02 + 0.955 + 0.955$$

$$= -0.12.$$

$$\Rightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \frac{1}{-0.12} \begin{pmatrix} 0.955 & 0.055 & -1 \\ 0.955 & -0.065 & -1 \\ -2.03 & 0.01 & 2 \end{pmatrix} \begin{pmatrix} 300 \\ 20 \\ 290 \end{pmatrix}$$

$$= \frac{1}{-0.12} \begin{pmatrix} -2.4 \\ -4.8 \\ -28.8 \end{pmatrix} = \begin{pmatrix} 20 \\ 40 \\ 240 \end{pmatrix} \quad /\!/$$

(6)(a) $\text{Mean} = \frac{1}{5}\int_0^5 \frac{15}{\sqrt{x^2+4x+3}}\,dx$

$$= \frac{1}{5}\int_0^5 \frac{15}{\sqrt{(x+2)^2-1}}\,dx$$

$$= 3\left[\cosh^{-1}(x+2)\right]_0^5$$

$$= 3\left(\ln\left(7+\sqrt{48}\right) - \ln\left(2+\sqrt{3}\right)\right)$$

$$= 3\ln\left(\frac{7+4\sqrt{3}}{2+\sqrt{3}}\right)$$

$$= 3\ln\left(2+\sqrt{3}\right) \quad //$$

(b) $\ln(2) - \frac{1}{3}\times 3\ln\left(2+\sqrt{3}\right)$

$$= \ln(2) - \ln\left(2+\sqrt{3}\right)$$

$$= \ln\frac{2}{2+\sqrt{3}} = \ln\left(4-2\sqrt{3}\right) \quad //$$

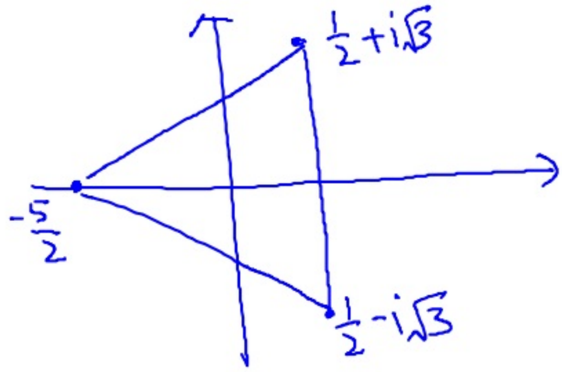
(7) (a) $\frac{1}{2} + i\sqrt{3}$ //

(b) $\left(\frac{1}{2} - i\sqrt{3}\right) + \left(\frac{1}{2} + i\sqrt{3}\right) + \alpha = -\frac{12}{8}$

$\Rightarrow 1 + \alpha = -\frac{3}{2}$

$\Rightarrow \alpha = -\frac{5}{2}$ //

(c)



(d) $\left|\left(\frac{1}{2} \pm i\sqrt{3}\right) - \left(-\frac{5}{2}\right)\right| = |3 \pm i\sqrt{3}| = \sqrt{12}$

$\left|\left(\frac{1}{2} + i\sqrt{3}\right) - \left(\frac{1}{2} - i\sqrt{3}\right)\right| = |i\, 2\sqrt{3}| = \sqrt{12}$

the three sides have equal length

$\Rightarrow$ equilateral triangle. //

(8)(a) $\left(1-\frac{1}{2}e^{5i\theta}\right)\left(1-\frac{1}{2}e^{-5i\theta}\right)$

$$= 1 - \frac{1}{2}e^{5i\theta} - \frac{1}{2}e^{-5i\theta} + \frac{1}{4}$$

$$= \frac{5}{4} - \cosh(5i\theta)$$

$$= \frac{5}{4} - \cos(5\theta) = \frac{1}{4}(5-4\cos(5\theta)) \quad //$$

(b) $S = e^{5i\theta} + \frac{1}{2}e^{10i\theta} + \frac{1}{4}e^{15i\theta} + \dots$

$$= \boxed{\frac{e^{5i\theta}}{1-\frac{1}{2}e^{5i\theta}}} = \frac{e^{5i\theta}\left(1-\frac{1}{2}e^{-5i\theta}\right)}{\left(1-\frac{1}{2}e^{5i\theta}\right)\left(1-\frac{1}{2}e^{-5i\theta}\right)}$$

answer

$$= \frac{e^{5i\theta}-\frac{1}{2}}{\frac{1}{4}(5-4\cos(5\theta))}$$

$$= \frac{4e^{5i\theta}-2}{5-4\cos(5\theta)}$$

(c) $\sum_{r=1}^{\infty} \frac{\sin(5r\theta)}{2^{r-1}} = \mathrm{Im} \sum_{r=1}^{\infty} \frac{e^{5r\theta i}}{2^{r-1}}$

$$= \mathrm{Im}\,\frac{4e^{5i\theta}-2}{5-4\cos(5\theta)} = \frac{4\sin(5\theta)}{5-4\cos(5\theta)} \quad //$$

(d) $\frac{4\cos(5\theta)-2}{5-4\cos(5\theta)} \quad //$

(9)(a) $\frac{d^2H}{dt^2} = -4\frac{dN}{dt} - 3\frac{dH}{dt} + 4$

$= -4(N+2H-t+1) - 3(-4N-3H+4t) + 4$

$= 8N + H - 8t$

$\therefore \frac{d^2H}{dt^2} + 2\frac{dH}{dt} + 5H$

$= (8N+H-8t) + 2(-4N-3H+4t) + (5H)$

$= (8N+H-8t) + (-8N+6H+8t) + (5H)$

$= 0$ //

(b) $H(t) = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$ //

(c) $\frac{dH}{dt} = e^{-t}\left(-2A\sin(2t) + 2B\cos(2t)\right)$

$+ e^{-t}\left(-A\cos(2t) - B\sin(2t)\right)$

$= e^{-t}\left((-A+2B)\cos(2t) + (-2A-B)\sin(2t)\right)$

$\Rightarrow N = \frac{1}{4}\left(-\frac{dH}{dt} - 3H + 4t\right)$

$= \frac{1}{4}\Big\{ e^{-t}\left((A-2B)\cos(2t) + (2A+B)\sin(2t)\right)$

$+ e^{-t}\left(-3A\cos(2t) - 3B\sin(2t)\right)$

$+ 4t \Big\}$

$= \frac{-A-B}{2}e^{-t}\cos(2t) + \frac{A-B}{2}e^{-t}\sin(2t) + t$ //

(d)(i) When $t=0$, $N=12$ and $H=430$

$\Rightarrow \frac{-A-B}{2} = 12$, $A = 430$

$\Rightarrow A = 430$, $B = -454$

$\therefore N(t) = 12e^{-t}\cos(2t) + 442e^{-t}\sin(2t) + t$

$H(t) = 430e^{-t}\cos(2t) - 454e^{-t}\sin(2t)$

$H(t) = 0 \Rightarrow 430\cos(2t) - 454\sin(2t) = 0$

$\Rightarrow \tan(2t) = \frac{430}{454}$

$\Rightarrow t = 0.37912775$

$\therefore$ the stalagmite reduces to nothing in 379 years (in year 2379) //

(ii) $N(0.075395) = 214.379$ million //

(iii) Population is far too large for a small stalagtite //