#Mock Papers

Year 13 Mock Set 02
Pure Paper 1

1. Given that a and b are positive constants, solve the simultaneous equations

$$ab = 25 \qquad \text{(I)}$$
$$\log_4 a - \log_4 b = 3 \qquad \text{(II)}$$

Show each step of your working, giving exact values for a and b.

(6)

$$\log_4 \frac{a}{b} = 3$$

$$\frac{a}{b} = 64$$

$$ab = 25$$

$$a^2 = 64(25)$$

$$a = 40$$

$$b = \frac{5}{8} \,//$$

2. A sequence is defined by

$$u_1 = 3$$
$$u_{n+1} = u_n - 5, \qquad n \geq 1$$

Find the values of

(a) u_2, u_3 and u_4 **(2)**

(b) u_{100} **(3)**

(c) $\sum_{i=1}^{100} u_i$ **(3)**

a)

$u_2 = -2$

$u_3 = -7$

$u_4 = -12$

b)

n^{th} term

$-5n+8$

100^{th} term

$-5(100)+8$

$= -492$

c) $\dfrac{(-492+3)100}{2} = -24450$

3. Find the first 3 terms in ascending powers of x of

$$\left(2 - \frac{x}{2}\right)^6$$

giving each term in its simplest form.

(4)

$$2^6 + 6(2)^5\left(\frac{-x}{2}\right) + \frac{6(5)}{2}(2)^4\left(\frac{-x}{2}\right)^2$$

$$= 64 - 96x + 60x^2 + \ldots \quad //$$

4. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

(a) Express $2\sin x - 4\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the exact value of R and the value of α, in radians, to 3 significant figures.

(3)

Erling Braut Haaland is a student in Norway. He records the number of hours of daylight every day for a year. He models the number of hours of daylight, H, by the continuous function given by the formula

$$H = 12 + 4\sin\left(\frac{2\pi t}{365}\right) - 8\cos\left(\frac{2\pi t}{365}\right), \qquad 0 \le t \le 365$$

where t is the number of days since he began recording.

(b) Using your answer to part(a), or otherwise, find the maximum and minimum number of hours of daylight given by this formula. Give your answer to 3 significant figures.

(3)

(c) Use the formula to find the values of t when $H = 17$, giving your answers to the nearest integer.

(6)

4a) $R\sin x\cos\alpha - R\cos x\sin\alpha$

$2\sin x - 4\cos x$

$\div$ $R\cos\alpha = 2$ (1)

$R\sin\alpha = 4$ (2)

$\tan\alpha = 2$

$\alpha = 1.11\ (3sf)$

$R = \sqrt{4^2+2^2}$

$R = \sqrt{20}$

$\sqrt{20}\sin(x - 1.11)$

b) $x = \frac{2\pi t}{365}$

$H = 12 + 4\sin x - 8\cos x$

$H = 12 + 2[2\sin x - 4\cos x]$

$H = 12 + 2[\sqrt{20}\sin(x-1.11)]$

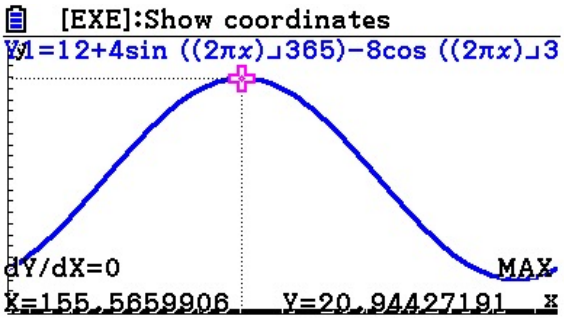
$H = 12 + 2\sqrt{20}\sin(x-1.11)$

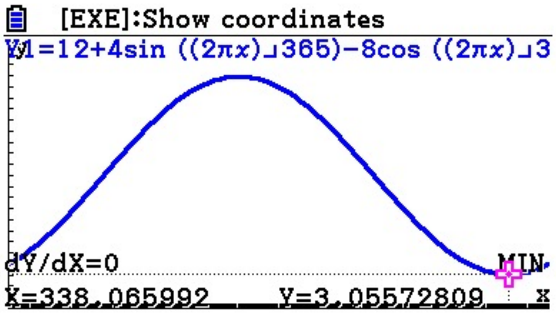
$-1 \le \sin(y) \le 1$

$-2\sqrt{20} \le 2\sqrt{20}\sin y \le 2\sqrt{20}$

$12 - 2\sqrt{20} \le 12 + 2\sqrt{20}\sin y \le 12 + 2\sqrt{20}$

$3.06 \le H \le 20.9$





c) $H = 12 + 2\sqrt{20}\sin(X - 1.11)$ $(H = 17)$

$17 = 12 + 2\sqrt{20}\sin(X - 1.11)$

$\sin(X - 1.11) = \frac{\sqrt{5}}{4}$ $\left(X = \frac{2\pi t}{365}\right)$

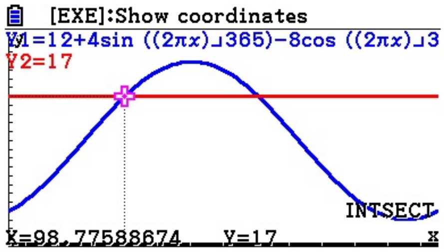
$0 \leq t \leq 365$

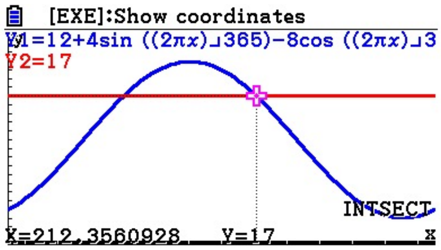
$0 \leq X \leq 2\pi$

$X - 1.11 = 0.593, 2.548$

$X = 1.703, 3.658$

$t = 99, 212$ (days)





5. The straight line l has equation $y = mx - 2$, where m is a constant.
The curve C has equation $y = 2x^2 + x + 6$
The line l does not cross or touch the curve C.

(a) Show that m satisfies the inequality

$$m^2 - 2m - 63 < 0$$

(3)

(b) Hence find the range of possible values of m

(4)

a) $mx - 2 = 2x^2 + x + 6$

$0 = 2x^2 + x(1-m) + 8$

$\Delta < 0$

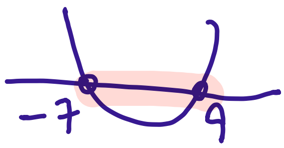
$b^2 - 4ac < 0$

$(1-m)^2 - 4(2)(8) < 0$

$m^2 - 2m + 1 - 64 < 0$

$m^2 - 2m - 63 < 0$

b) $(m-9)(m+7) < 0$



$-7 < m < 9$

6. (a) Find $\int \frac{4x+3}{x}\,dx, \qquad x > 0$

(2)

(b) Given that $y = 25$ at $x = 1$, solve the differential equation

$$\frac{dy}{dx} = \frac{(4x+3)\,y^{\frac{1}{2}}}{x} \qquad x > 0, y > 0$$

giving your answer in the form $y = \left[g(x)\right]^2$.

(5)

a) $\int 4 + \frac{3}{x}\,dx$

$= 4x + 3\ln x + C$

b) $\int \frac{4x+3}{x}\,dx = \int y^{-\frac{1}{2}}\,dy$

$4x + 3\ln x = 2y^{\frac{1}{2}} + C$ $\quad \{x=1, y=25\}$

General solution

$4 = 2(5) + C$

$C = -6$

$4x + 3\ln x = 2y^{\frac{1}{2}} - 6$

particular solution

$4x + 3\ln x + 6 = 2\sqrt{y}$

$\left(2x + \frac{3}{2}\ln x + 3\right)^2 = y$ $\Leftarrow$ $\boxed{\text{answer}}$

NB: $g(x) = 2x + \frac{3}{2}\ln x + 3$

7. The point P with coordinates $\left(\frac{\pi}{2}, 1\right)$ lies on the curve with equation

$$4x \sin x = \pi y^2 + 2x, \qquad \boxed{\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}}$$

Find an equation of the normal to the curve at P.

(6)

usually we need "exact form"

domain of x

$\Rightarrow$ it gives the "sign" of $\sin x$ / $\cos x$ etc

$\Rightarrow$ diff both sides wrt x:

$$\underbrace{4x[\cos x] + [\sin x]4}_{\text{product rule}} = \underbrace{2\pi y \frac{dy}{dx}}_{\text{implicit diff}} + 2$$

$$\frac{4x\cos x + 4\sin x - 2}{2\pi y} = \frac{dy}{dx} \qquad \{x = \tfrac{\pi}{2},\ y = 1\}$$

$$\frac{4(\frac{\pi}{2})(0) + 4\sin(\frac{\pi}{2}) - 2}{2\pi(1)} = \frac{2}{2\pi} = \frac{1}{\pi} \quad (\text{gradient of tangent})$$

$m_{normal} = -\pi$

eqt of normal

$$y - 1 = -\pi\left(x - \frac{\pi}{2}\right)$$

$$y = -\pi x + \frac{\pi^2}{2} + 1$$

"any form" since Q did not specify.

8. (Note that for Edexcel questions, (i) and (ii) within a single question usually means they are two unrelated questions, they are only combined into a question since they are the same "Chapter".)

(i) Differentiate $y = 5x^2 \ln 3x, \qquad x > 0$

(2)

(ii) Given that

$$y = \frac{x}{\sin x + \cos x}, \qquad -\frac{\pi}{4} < x < \frac{3\pi}{4}$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(1+x)\sin x + (1-x)\cos x}{1 + \sin 2x}, \qquad -\frac{\pi}{4} < x < \frac{3\pi}{4}$$

(4)

i) $5x^2\left[\frac{3}{3x}\right] + \ln 3x\left[10x\right]$

ii) $\dfrac{(\sin x + \cos x)(1) - (x)(\cos x - \sin x)}{(\sin x + \cos x)^2}$ { Quotient Rule }

$= \dfrac{\sin x + \cos x - x\cos x + x\sin x}{\sin^2 x + \cos^2 x + 2\sin x\cos x} = \dfrac{\sin x(1+x) + \cos x(1-x)}{1 + \sin 2x}$

//

9. Use the substitution $x = 2\sin\theta$ to find the exact value of

$$\int_0^{\sqrt{3}} \frac{1}{(4-x^2)^{\frac{3}{2}}}\,\mathrm{d}x$$

(7)

$x = \sqrt{3}$ $\quad \sqrt{3} = 2\sin\theta$

$\theta = \frac{\pi}{3}$

$x = 0$ $\quad 0 = 2\sin\theta$

$\theta = 0$

$x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$

$dx = 2\cos\theta\, d\theta$

$x = 2\sin\theta$

$4 - x^2 = 4 - 4\sin^2\theta$

$= \underline{4\cos^2\theta}$

$\Rightarrow \int_0^{\frac{\pi}{3}} \frac{1}{(4\cos^2\theta)^{\frac{3}{2}}}\, 2\cos\theta\, d\theta$

$\int_0^{\frac{\pi}{3}} \frac{1}{4}\cos^{-2}\theta\, d\theta$

$\int_0^{\frac{\pi}{3}} \frac{1}{4}\sec^2\theta\, d\theta$

$\frac{1}{4}\left[\tan\theta\right]_0^{\frac{\pi}{3}}$

$= \frac{\sqrt{3}}{4}$ //

10. **In this question you must show all stages of your working.**
Solutions relying on calculator technology are not acceptable.

(a) Prove that

$$\frac{1-\cos 2x}{1+\cos 2x} \equiv \tan^2 x \qquad x \neq (2n+1)\,90°, n \in \mathbb{Z}$$

(3)

(b) Hence, or otherwise, solve, for $-90° < \theta < 90°$,

$$\frac{2-2\cos 2\theta}{1+\cos 2\theta} - 2 = 7\sec\theta$$

Give your answer in degrees to one decimal place.

(6)

a) $\dfrac{1-(\cos^2 x-\sin^2 x)}{1+(\cos^2 x - \sin^2 x)} = \dfrac{\sin^2 x + \cos^2 x - \cos^2 x + \sin^2 x}{\sin^2 x + \cos^2 x + \cos^2 x - \sin^2 x}$

$= \dfrac{2\sin^2 x}{2\cos^2 x}$

$= \underline{\underline{\tan^2 x}}$

b) $2\tan^2 x - 2 = 7\sec\theta$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$2(\sec^2 x - 1) - 2 = 7\sec\theta$

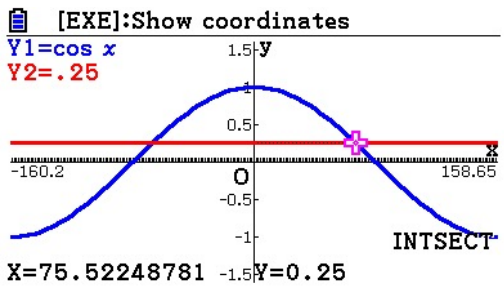
$2\sec^2\theta - 7\sec\theta - 4 = 0$

$(2\sec\theta + 1)(\sec\theta - 4) = 0$

$\sec\theta = 4 \qquad \sec\theta = \frac{-1}{2}$

$\cos\theta = \frac{1}{4} \qquad \cos\theta = -2$ (reJ)

$\theta = 75.5, -75.5$ //



11.

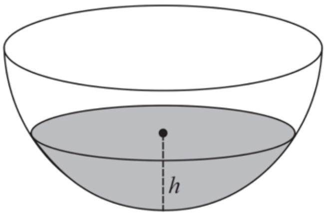


Figure 1

Figure 1 shows a mixing bowl.
Water is flowing into the bowl at a rate of 180 $cm^3 s^{-1}$.
When the height of the water is h cm, the volume of water V cm^3 is given by

$$V = \frac{1}{3}\pi h^2(90 - h) \qquad 0 \le h \le 30$$

Find the rate of change of the height of water, in $cm\,s^{-1}$, when $h = 15$.
Give your answer to 2 significant figures.

(5)

$\frac{dv}{dt} = 180$

$V = \frac{1}{3}\pi h^2(90-h)$

$\frac{dv}{dh} = [\frac{1}{3}\pi h^2][-1] + [90-h][\frac{2}{3}\pi h]$

$\frac{dv}{dh} = -\frac{1}{3}\pi h^2 + 60\pi h - \frac{2}{3}\pi h^2$

need $\frac{dh}{dt}$ $\Rightarrow$

$\frac{dh}{dt} = \frac{dh}{dv} \cdot \frac{dv}{dt}$ when $h=15$,

$= \frac{180}{-\frac{1}{3}\pi(15)^2 + 60\pi(15) - \frac{2}{3}\pi(15)^2}$

$\frac{dh}{dt} = 0.085$ (2sf) $cm\,s^{-1}$ //

Figure 2 shows the design for a "Head Boy" badge.
The design consists of two congruent triangles, AOC and BOC, joined to a sector AOB of a circle with centre O.
Given that

- $\angle AOB = \alpha$
- $AO = OB = 3$ cm
- $OC = 5$ cm
- the area of sector $AOB = 7.2$ cm^2

(a) show that $\alpha = 1.6$ radians. **(2)**

$\frac{\alpha}{2\pi}(\pi r^2) = \text{Area}$

$\frac{\alpha}{2} r^2 = 7.2$

$\alpha = \frac{7.2(2)}{9}$

$\alpha = 1.6$ radians //

(b) Hence find

(i) the area of the badge, giving your answer in cm^2 to two significant figures.

(ii) the perimeter of the badge, giving your answer in cm to one decimal place. **(8)**

12.

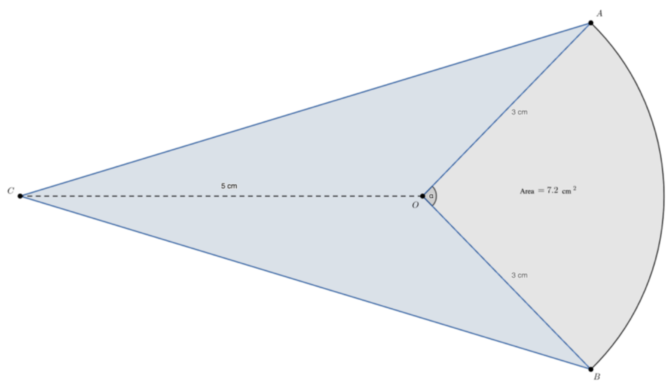


Figure 2

$\angle COA = \frac{2\pi - 1.6}{2} = 2.342$ rad

Area of $\Delta = \frac{1}{2}(3)(5)\sin(2.342)$

Area of Badge $=$

$7.2 + 2(5.38\ldots)$

$= 17.96\ldots$

≈ 18 cm^2 (2sf)

b(i) $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 5^2 + 3^2 - 2(5)(3)\cos(2.342)$

$c^2 = 54.9\ldots$

$c = 7.41\ldots$

perimeter $= 7.41 + 7.41 + \frac{\alpha}{2\pi} \cdot 2\pi r$

$= 19.6$ cm //

13. Given that n is an integer, use algebra, to prove by contradiction, that

$$\text{if } n^3 \text{ is even then } n \text{ is even.}$$

(4)

assume
if n^3 is even then n is odd

$\Rightarrow n = 2k+1$

$n^3 = (2k+1)^3$

$= (2k)^3 + 3(2k)^2 + 3(2k) + 1$

$= 8k^3 + 12k^2 + 6k + 1$

$\Rightarrow 2(4k^3 + 6k^2 + 3k) + 1$

even + 1

$\Rightarrow$ odd

contradiction, if n^3 is even, n cannot be odd,
therefore
if n^3 is even, n must be even. //

14.

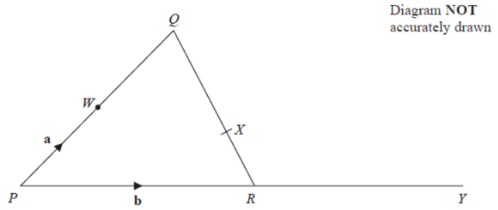


Figure 3

As shown in Figure 3,

- PQR is a triangle.
- The midpoint of PQ is W.
- X is the point on QR such that $QX : XR = 2 : 1$
- PRY is a straight line.
- $\overrightarrow{PW} = \mathbf{a}$
- $\overrightarrow{PR} = \mathbf{b}$

(a) Find, in terms of **a** and **b**

(i) $\overrightarrow{QR}$

(ii) $\overrightarrow{QX}$

(iii) $\overrightarrow{WX}$

(3)

Given that R is the midpoint of the straight line PRY

(b) Use a vector method to show that WXY is a straight line.

(2)

$\overrightarrow{PQ} = 2a$

$\overrightarrow{QR} = \overrightarrow{QP} + \overrightarrow{PR}$

$= -2a + b$

$\overrightarrow{QX} = \frac{2}{3}(-2a + b)$

$\overrightarrow{WX} = \overrightarrow{WQ} + \overrightarrow{QX}$

$= a + \frac{-4}{3}a + \frac{2}{3}b$

$= \frac{-1}{3}a + \frac{2}{3}b$

$= \frac{1}{3}(-a + 2b)$

$\overrightarrow{XY} = \overrightarrow{XR} + \overrightarrow{RY}$

$= \frac{1}{3}(-2a + b) + b$

$= \frac{-2}{3}a + \frac{4}{3}b$

$= \frac{2}{3}(-a + 2b)$

$2\overrightarrow{WX} = \overrightarrow{XY}$

therefore $WX \parallel XY$

and they share a point X

$\therefore$ WXY is a straight line.

15. The function f is defined by

$$\mathrm{f}(x) = x^2 + 4x + 1 \qquad x \in \mathbb{R}$$

(a) Find the range of $\mathrm{f}(x)$.

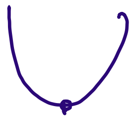
(3)

(b) Explain why the function f does not have an inverse.

(1)

a) $y = (x+2)^2 - 4 + 1$

$y = (x+2)^2 - 3$



$(-2,3)$

range : $f(x) \geqslant 3$

b)

f is a many to one function.