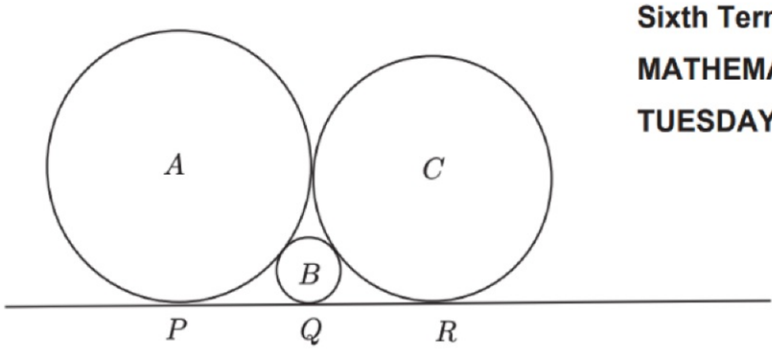
Sixth Term Examination Papers

MATHEMATICS 1

TUESDAY 14 JUNE 2016

5 **(i)**



The diagram shows three touching circles A, B and C, with a common tangent PQR. The radii of the circles are a, b and c, respectively.

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.

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SoHokMaths by A. Chan
STEP 1 June 2016 Q5

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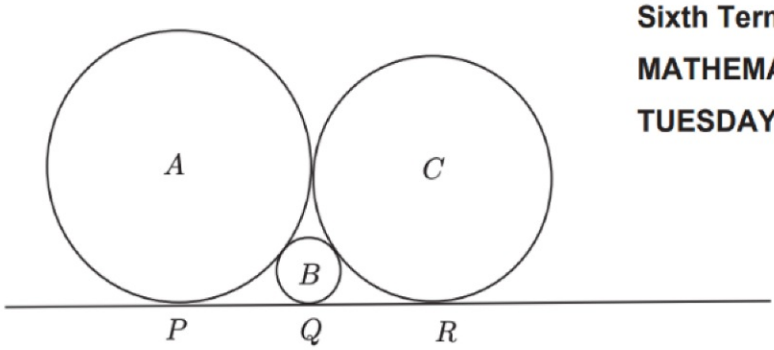

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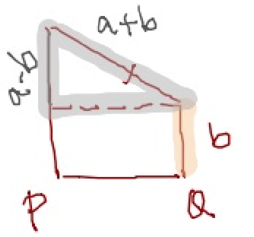

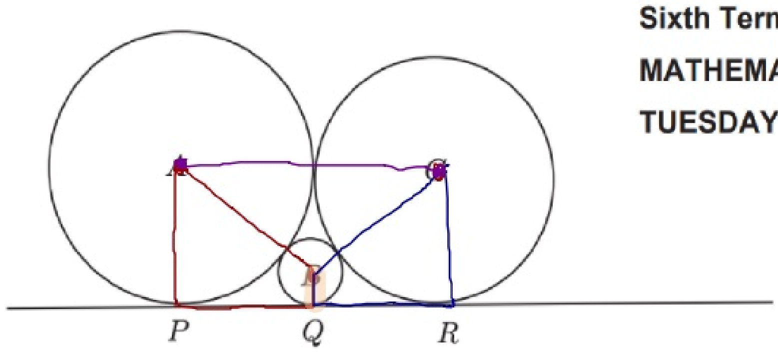

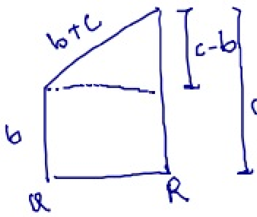

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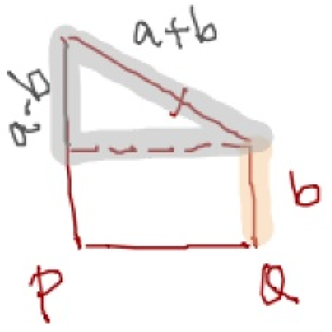
$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2. \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.



$PQ^2 + (a-b)^2 = (a+b)^2$

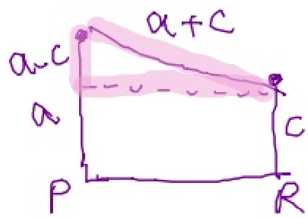
$PQ^2 = (a+b)^2 - (a-b)^2$

$= (a^2+b^2+2ab) - (a^2+b^2-2ab)$

$PQ^2 = 4ab$

$PQ = \sqrt{4ab}$

$= 2\sqrt{ab}$



Similarly

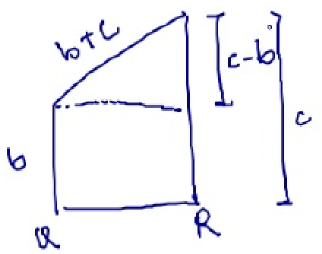
$(b+c)^2 = QR^2 + (c-b)^2$

$\Rightarrow QR^2 = 4bc$

$QR = 2\sqrt{bc}$

Note

$(a-c)^2 = (c-a)^2$



Also,

$(a+c)^2 = PR^2 + (a-c)^2$

$PR^2 = 4ac$

$PR = +2\sqrt{ac}$

$PR = PQ + QR$

$2\sqrt{ac} = 2\sqrt{bc} + 2\sqrt{ab}$

$\sqrt{ac} = \sqrt{bc} + \sqrt{ab}$

$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

as required

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad \text{condition} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2. \qquad (**)$$

$(a+b)^2 = a^2+b^2+2ab$

Identity

$(a+b+c)^2 = (a+b)^2 + 2(a+b)c + c^2$

$= a^2+b^2+c^2+2(ab+ac+bc)$

$RHS = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$

$= \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)$

$\Rightarrow LHS = 2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$

Note

$\left[\frac{1}{b}\right]^2 = \left(\frac{1}{a}+\frac{1}{c}+\frac{2}{\sqrt{ac}}\right)^2$

$\frac{1}{b^2} = \frac{1}{a^2}+\frac{1}{c^2}+\frac{4}{ac}+2\left(\frac{1}{ac}+\frac{2}{a\sqrt{ac}}+\frac{2}{c\sqrt{ac}}\right)$

$\Rightarrow \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

$\frac{1}{b} = \frac{1}{a}+\frac{1}{c}+2\left(\frac{1}{\sqrt{ac}}\right)$

$\Rightarrow RHS = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$

$= \left(\frac{1}{a}+\frac{1}{a}+\frac{1}{c}+\frac{1}{c}+\frac{2}{\sqrt{ac}}\right)^2$

$= \left(\frac{2}{a}+\frac{2}{c}+\frac{2}{\sqrt{ac}}\right)^2$

$= \frac{4}{a^2}+\frac{4}{c^2}+\frac{4}{ac}+2\left(\frac{4}{ac}+\frac{4}{a\sqrt{ac}}+\frac{4}{c\sqrt{ac}}\right)$

$= \frac{4}{a^2}+\frac{4}{c^2}+\frac{12}{ac}+\frac{8}{a\sqrt{ac}}+\frac{8}{c\sqrt{ac}}$

$LHS = 2\left[\frac{1}{a^2}+\frac{1}{c^2}+\frac{1}{a^2}+\frac{1}{c^2}+\frac{4}{ac}+\frac{2}{ac}+\frac{4}{a\sqrt{ac}}+\frac{4}{c\sqrt{ac}}\right]$

$= \frac{4}{a^2}+\frac{4}{c^2}+\frac{12}{ac}+\frac{8}{a\sqrt{ac}}+\frac{8}{c\sqrt{ac}}$

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad \text{aim } \boxed{\beta = \alpha + r} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \qquad \alpha + r - \beta = 0 \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.

$(**)$

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$$

$$\boxed{\begin{aligned} RHS &= \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \\ &= \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)\end{aligned}}$$

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) \rightarrow \underline{\text{to show}} \quad \text{if } b<c<a \text{ then } (*)$$

try: $\alpha = \frac{1}{\sqrt{a}} \quad \beta = \frac{1}{\sqrt{b}} \quad r = \frac{1}{\sqrt{c}}$

$$\alpha^4+\beta^4+r^4 = 2(\alpha^2\beta^2+\alpha^2r^2+\beta^2r^2)$$

<u>identity</u>

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac)$$

$$(\alpha^2+\beta^2+r^2)^2 = \alpha^4+\beta^4+r^4+2(\alpha^2\beta^2+\alpha^2r^2+\beta^2r^2)$$

$$\alpha^4+\beta^4+r^4-2(\alpha^2\beta^2+\alpha^2r^2+\beta^2r^2)=0 \qquad (**) \text{ (starting statement)}$$

$$(\alpha^2+r^2-\beta^2)^2 = (\alpha^2+r^2)^2+(-\beta^2)^2-2(\alpha^2+r^2)(\beta^2)$$

$$= \alpha^4+r^4+2\alpha^2r^2+\beta^4-2\alpha^2\beta^2-2\beta^2r^2 \quad //$$

$$\rightarrow (\alpha^2+r^2-\beta^2)^2-2\alpha^2r^2+2\alpha^2\beta^2+2\beta^2r^2-2\alpha^2\beta^2-2\alpha^2r^2-2\beta^2r^2=0$$

$$(\alpha^2+r^2-\beta^2)^2 = 4\alpha^2r^2$$

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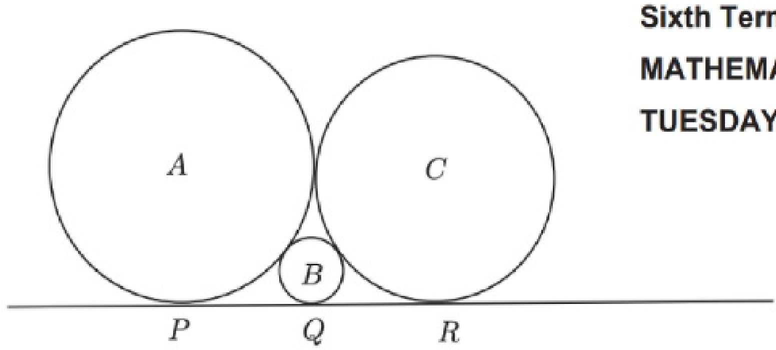

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$(\alpha^2+r^2-\beta^2)^2 = 4\alpha^2 r^2$

Small Big

$b < c < a$

$\frac{1}{b} > \frac{1}{c} > \frac{1}{a}$

$\frac{1}{\sqrt{b}} > \frac{1}{\sqrt{c}} > \frac{1}{\sqrt{a}}$

$\boxed{\beta = \alpha + r}$

$\boxed{\beta > r > \alpha}$

$\alpha^2+r^2-\beta^2 = \pm 2\alpha r$

$\alpha^2+r^2-\beta^2 = 2\alpha r$ OR $\alpha^2+r^2-\beta^2 = -2\alpha r$

$\alpha^2+2\alpha r+r^2 = \beta^2$ OR $\alpha^2-2\alpha r+r^2 = \beta^2$

$(\alpha+r)^2 = \beta^2$ OR $(\alpha-r)^2 = \beta^2$

$\alpha+r = \pm\beta$ OR $\alpha - r = \pm\beta$

+ + +

$\alpha+r=\beta$ ✓

OR

+ + -ve

$\alpha+r=-\beta$ X

OR

+ + Biggest

$\alpha-r=\beta$ X

OR

+ve smallest, + smaller, -ve smallest

$\alpha-r=-\beta$ X

$\Rightarrow (*)$

□