13Fm Core Pure Maths Mock Paper 2 2021.05.19

This exam has 9 questions, for a total of 75 marks.

- Print in "booklets" will allow all questions to be on the left hand side.
- If instead you print in 2-in-1 settings, print the second page up to the last page first, then print the first page separately.

Question	Marks	Score
1	6	
2	6	
3	5	
4	8	
5	7	
6	8	
7	8	
8	10	
9	17	
Total:	75	

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1. (a) Using hyperbolic identities, which should be clearly stated, prove that

$$\frac{1+\tanh^2 x}{1-\tanh^2 x} \equiv \cosh 2x$$

(4)

(b) Solve the equation

$$\frac{1+\tanh^2 x}{1-\tanh^2 x} = \frac{17}{8}$$

giving each answer in exact form.

(2)

Question 1 continued

(Total for Question 1 is 6 marks)

2.

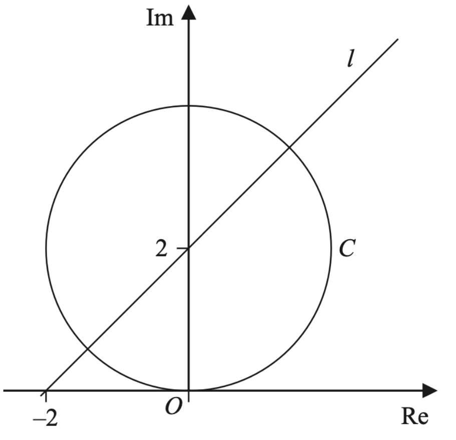

Figure 1

The Argand diagram, shown in Figure 1, shows a circle C and a half-line l.

(a) Write down the equation of the locus of points represented in the complex plane by

(i) the circle C,

(ii) the half-line l.

(2)

(b) Use set notation to describe the sets of points that lie on both C and l.

(1)

(c) Find the complex numbers that lie on both C and l, giving your answers in the form $a + \mathrm{i}b$, where a, $b \in \mathbb{R}$

(3)

Question 2 continued

Question 2 continued

Question 2 continued

(Total for Question 2 is 6 marks)

3.

$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The matrices $\mathbf{P}$ and $\mathbf{Q}$ represent linear transformations, P and Q respectively, of the plane.

The linear transformation M is formed by first applying P and then applying Q.

(a) Find the matrix $\mathbf{M}$ that represents the linear transformation M.

(2)

(b) Show that the invariant points of the linear transformation M form a line in the plane, stating the equation of this line.

(3)

Question 3 continued

Question 3 continued

Question 3 continued

(Total for Question 3 is 5 marks)

4. (a) Use the standard results of $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} r(r+1) = \frac{n}{3}(n+a)(n+b)$$

where a and b are integers to be determined.

(4)

(b) Hence show that

$$\log 9 + 2\log 27 + 3\log 81 + \cdots + 11\log(531441)$$

can be written in the form $p\log q$, where p and q are integers to be determined.

(4)

Question 4 continued

Question 4 continued

Question 4 continued

(Total for Question 4 is 8 marks)

5. A village cheese-monger, Sharjil, sells three kinds of cheese: Airedale, Brie and Cheddar. During 2015 he sold a total of 300 kg of cheese, and he sold 20 kg more Brie than Airedale. During 2016:

- his sales of Airedale increased by 2%
- his sales of Brie increased by 1%
- his sales of Cheddar decreased by 4.5%
- overall he sold 10 kg less cheese than in 2015

Form and solve a matrix equation to determine how much of each type of cheese Sharjil sold in 2015.

(7)

Question 5 continued

Question 5 continued

Question 5 continued

(Total for Question 5 is 7 marks)

6.

$$f(x) = \frac{15}{\sqrt{x^2 + 4x + 3}} \qquad x > -1$$

(a) Find the mean value of $f(x)$ over the interval $[0, 5]$.

Give your answer in the form $a\ln(b + \sqrt{3})$, where a and b are integers to be determined.

(6)

(b) Hence find the mean value of $\ln 2 - \frac{1}{3}f(x)$ over the interval $[0, 5]$.

Give your answer as a single logarithm in its simplest form.

(2)

Question 6 continued

Question 6 continued

Question 6 continued

(Total for Question 6 is 8 marks)

7.

$$f(z) = 8z^3 + 12z^2 + 6z + 65$$

Given that $\dfrac{1}{2} - i\sqrt{3}$ is a root of the equation $f(z) = 0$

(a) write down the other complex root of the equation,

(1)

(b) use algebra to solve the equation $f(z) = 0$ completely.

(3)

(c) Show the roots of $f(z)$ on a single Argand diagram.

(2)

(d) Show that the roots of $f(z)$ form the vertices of an equilateral triangle in the complex plane.

(2)

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 8 marks)

8. (a) Show that $\left(1 - \frac{1}{2}e^{5i\theta}\right)\left(1 - \frac{1}{2}e^{-5i\theta}\right) \equiv \frac{1}{4}(5 - 4\cos 5\theta)$ **(3)**

(b) Given that the series

$$S = e^{5i\theta} + \frac{1}{2}e^{10i\theta} + \frac{1}{4}e^{15i\theta} + \frac{1}{8}e^{20i\theta} + \cdots$$

is convergent, express S in terms of $e^{5i\theta}$ **(2)**

(c) Hence show that $\displaystyle\sum_{r=1}^{\infty} \frac{\sin(5r\theta)}{2^{r-1}} = \frac{4\sin 5\theta}{5 - 4\cos 5\theta}$ **(4)**

(d) Write down a similar expression for $\displaystyle\sum_{r=1}^{\infty} \frac{\cos(5r\theta)}{2^{r-1}}$ **(1)**

Question 8 continued

Question 8 continued

Question 8 continued

(Total for Question 8 is 10 marks)

9.

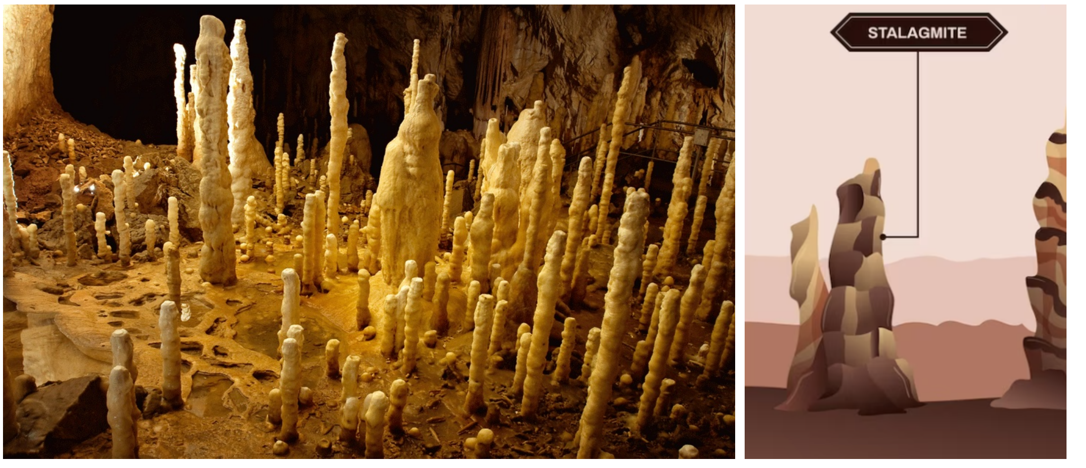

Figure 2

A stalagmite is an upward-growing mound of mineral deposits, as shown in Figure 2.

A scientist, Jerick, thinks that N, the number of bacteria in the colony in millions, and the height, H, of the stalagmite in centimetres, can be modelled by the equations:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N + 2H - t + 1$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -4N - 3H + 4t$$

where t is measured in thousands of years since the start of the year 2000.

(a) Show that $\dfrac{\mathrm{d}^2H}{\mathrm{d}t^2} + 2\dfrac{\mathrm{d}H}{\mathrm{d}t} + 5H = 0$

(3)

(b) Find a general solution, in terms of t, for the height of the stalagmite in centimetres.

(4)

(c) Hence find a general solution, in terms of t, for the number of bacteria in the colony in millions.

(3)

Jerick estimates that at the start of the year 2000, there were 12 million bacteria in the colony and that the stalagmite was 4.3m tall.

(a) (i) According to Jerick's model, in what year will the stalagmite have reduced to nothing?

(5)

(ii) How many bacteria will there be in the colony at this time, according to the model?

(1)

(iii) Using your answer to part (ii), comment on the suitability of the model.

(1)

Question 9 continued

Question 9 continued

Question 9 continued

Question 9 continued

(Total for Question 9 is 17 marks)

Total for paper is 75 marks