1. **Answer ALL questions. Write your answers in the spaces provided.**

(a) Find

$$\int \frac{1}{x^2 + 6x + 25}\,dx$$

(3)

(b) Hence find the exact value of

$$\int_{-3}^{1} \left(1 - \frac{25}{x^2 + 6x + 25}\right) dx$$

giving the answer in simplest form.

(3)

A student claims that the magnitude of the answer to part (b) gives the total area bounded by the curve $y = 1 - \dfrac{25}{x^2 + 6x + 25}$ and the x-axis between the line $x = -3$ and the line $x = 1$

(c) State, with a reason, whether or not the student is correct.

(2)

a) $\displaystyle\int \frac{1}{(x+3)^2+16}\,dx$

$= \dfrac{1}{4}\arctan\left(\dfrac{x+3}{4}\right) + c$

b) $\left[x - \dfrac{25}{4}\arctan\left(\dfrac{x+3}{4}\right)\right]_{-3}^{1}$

$4 - \dfrac{25}{4}\arctan 1 \; \cancel{+\dfrac{25}{4}\arctan 0}$

$4 - \dfrac{25\pi}{16}$.

c) A portion of the graph is below the x axis. This means that the integral we evaluated will have a 'negative' area which will cancel a portion of the positive area, ∴ no, not correct.

2. A company operating a coal mine is concerned about the mine running out of coal. It is estimated that 2.5 million tonnes of coal are left in the mine. The company wishes to mine all of this coal in 20 years.

In order to mine the coal in a regulated manner, the company models the amount of coal to be mined in the coming years by the formula

$$M_r = \frac{10}{r^2 + 8r + 15}$$

where M_r is the amount of coal, in millions of tonnes, mined in year r, with the first year being year 1

(a) Show that, according to the model, the total amount of coal, in millions of tonnes, mined in the first n years is given by

$$T_n = \frac{9n^2 + 41n}{k(n+4)(n+5)}$$

where k is a constant to be determined.

(6)

(b) Explain why, according to this model, the mine will never run out of coal.

(2)

The company decides to mine an extra fixed amount each year so that all the coal will be mined in exactly 20 years.

(c) Refine the formula for M_r so that 2.5 million tonnes of coal will be exhausted in exactly 20 years of mining.

(2)

$$M_r = \frac{A}{r+3} + \frac{B}{r+5} = \frac{5}{r+3} - \frac{5}{r+5}$$

$$5\sum_{1}^{n} \frac{1}{r+3} - \frac{1}{r+5}$$

$r=1$ $\quad \frac{1}{4} - \cancel{\frac{1}{6}}$ $\qquad r=n-1$ $\quad \cancel{\frac{1}{n+2}} + \frac{1}{n+4}$

$r=2$ $\quad \frac{1}{5} - \cancel{\frac{1}{7}}$ $\qquad r=n$ $\quad \cancel{\frac{1}{n+3}} + \frac{1}{n+5}$

$r=3$ $\quad \cancel{\frac{1}{6}} - \cancel{\frac{1}{8}}$

Question 2 continued

$$5\left(\frac{1}{4}+\frac{1}{5}-\frac{1}{n+4}-\frac{1}{n+5}\right)$$

$$5\left[\frac{9}{20}-\frac{1}{n+4}-\frac{1}{n+5}\right]$$

$$\cancel{5}\left[\frac{9(n+4)(n+5)-20(n+5)-20(n+4)}{4\,\cancel{20}(n+4)(n+5)}\right]$$

$$\frac{9n^2+41n}{4(n+4)(n+5)} \quad //$$

b) $T_n=\dfrac{9n^2+41n}{4n^2+36n+80} = \dfrac{9+\frac{41}{n}}{4+\frac{36}{n}+\frac{80}{n^2}}$

as $n\to\infty$, $T_n\to\frac{9}{4}$

$\frac{9}{4}<2.5$, $\therefore$ there will always be coal left as 2.25 million tonnes will be the limiting value.

c) $T_{20}=\dfrac{221}{120}$

$2.5-\dfrac{221}{120}=\dfrac{79}{120}$, $\dfrac{79}{120}\div 20=\dfrac{79}{2400}$

$M_r=\dfrac{79}{2400}+\dfrac{10}{r^2+8r+15}$

(Total for Question 2 is 10 marks)

3. (a) Find, in terms of the real constant k, the determinant of the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & k & 2 \end{pmatrix}$$

(2)

Three distinct planes, Π_1, Π_2 and Π_3, are defined by the equations

$$\Pi_1 \quad \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 4$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\Pi_3 : x + ky + 2z = -1$$

where λ and μ are scalar parameters.

(b) Find an equation in Cartesian form for

(i) Π_1

(ii) Π_2

(4)

Given that the three planes Π_1, Π_2 and Π_3 form a sheaf,

(c) use the answer to part (a) to explain why $k = -1$

(2)

a) $1\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} - k\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$

$-5 + 5k + 10 = 5k + 5$

i) $3x + 2y + z = 4$

ii) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

$\mathbf{r} \cdot \begin{pmatrix} -2 \\ -3 \\ +1 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

~~$x + 2y + 3z = -5$~~

$-2x - 3y + z = -5$

$2x + 3y - z = 5$

c) planes meet when

$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$

$\therefore \det M \neq 0$ if no unique solution exists,

so $k \neq -1$.

4. (i) Prove by induction that, for $n \in \mathbb{N}$,

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^n = \begin{pmatrix} 3n+1 & -n \\ 9n & 1-3n \end{pmatrix}$$

(6)

(ii) Consider the statement

$$n^2 < 2^n \qquad \text{for all } n \in \mathbb{Z}^+$$

A student attempts to prove this statement using induction as follows.

Student's response

For $n = 1$ we have $1^2 = 1$ and $2^1 = 2$
Since $1 < 2$ the statement is true for $n = 1$

Suppose it is true for $n = k$, so $k^2 < 2^k$

Line 4 → Then $(k+1)^2 = k^2 + 2k + 1 < k^2 + k^2$ (since $2k + 1 < k^2$ for $k \in \mathbb{Z}^+$)
$= 2k^2$
$< 2 \times 2^k$ (by the assumption $k^2 < 2^k$)
$= 2^{k+1}$

Hence the result is true for $n = k + 1$

So the result is true for $n = 1$ and if it is true for $n = k$ then it is true for $n = k + 1$, and hence it is true for all positive integers n by mathematical induction.

(a) Show by a counterexample that the statement is not true.

Given that the only mathematical error in the student's proof occurs in line 4,

(b) identify the error made in the student's proof,

(c) hence determine for which positive integers the statement is true, explaining your reasoning.

(5)

Basis: $n = 1$

LHS: $\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^1 = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$ RHS: $\begin{pmatrix} 3+1 & -1 \\ 9 & 1-3 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$

true for $n = 1$.

Assume true for $n = k$

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^k = \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}$$

Question 4 continued

Prove true for $n=k+1$

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^{k+1} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^{k} \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix} \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 12k+4-9k & -3k-1+2k \\ 36k+4-12k & -9k-2+6k \end{pmatrix}$$

$$= \begin{pmatrix} 3(k+1)+1 & -(k+1) \\ 9(k+1) & 1-3(k+1) \end{pmatrix}$$

$\therefore$ True for $n=k+1$.

if true for $n=k$, then true for $n=k+1$. Since true for $n=1$, true for all $n \in \mathbb{N}$.

a) $2^2=4$, $2^2=4$

$4 \nless 4$.

~~b)~~ c) ~~the~~ $2k+1<k^2$ is not true for all $k \in \mathbb{Z}^+$

It is true for $k \geq 3$

$\therefore$ the base case must be for n where ~~then~~ $n \geq 3$

However, $n^2 < 2^n$ only holds if $n \geq 5$.

$\therefore$ true for $n=1$ and $n \geq 5$, 5.

$$y = \arctan(\sinh(x))$$

(a) Show that $\dfrac{d^3y}{dx^3} = \dfrac{dy}{dx} - 2\left(\dfrac{dy}{dx}\right)^3$

(7)

(b) Hence find $\dfrac{d^5y}{dx^5}$ in terms of $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$

(4)

(c) Find the Maclaurin series for y, in ascending powers of x, up to and including the term in x^5

(3)

$$\frac{dy}{dx} = \frac{\cosh x}{1+\sinh^2 x} = \frac{1}{\cosh x} = \operatorname{sech} x$$

$$\frac{d^2y}{dx^2} = -\operatorname{sech} x \tanh x$$

$$\frac{d^3y}{dx^3} = \operatorname{sech} x \tanh^2 x - \operatorname{sech}^3 x$$

$$= \operatorname{sech} x(1-\operatorname{sech}^2 x) - \operatorname{sech}^3 x$$

$$= \operatorname{sech} x - 2\operatorname{sech}^3 x$$

$$\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^3$$

b) $$\frac{d^4y}{dx^4} = \frac{d^2y}{dx^2} - 6\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2}$$

$$\frac{d^5y}{dx^5} = \frac{d^3y}{dx^3} - 12\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)^2 - 6\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}$$

c) $$f(x) = x - \frac{x^3}{3!} + \frac{5x^5}{5!}$$

$$= x - \frac{x^3}{6} + \frac{x^5}{24}$$

6.

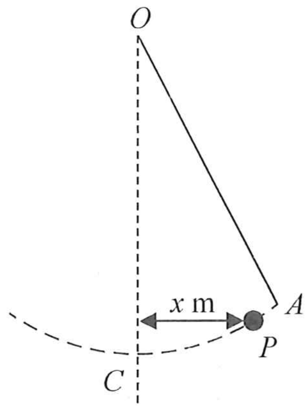


Figure 2

A child plays on a rope swing.
One end of the rope is attached to a tree and the child sits on a large knot at the other end of the rope.
The child swings back and forth in a vertical plane.

The rope is modelled as a light and inextensible string. The child is modelled as a particle.

Figure 2 represents the child and the rope swing. The rope is attached to the tree at the point O and the point C is vertically below O. The point P represents the child.

The horizontal displacement of P from the line OC at time t seconds ($t \geqslant 0$) is x metres, as shown in Figure 2.

The motion of P is modelled by the differential equation

$$\ddot{x} + 2\dot{x} + \lambda x = 0$$

where λ is a positive constant.

The child is initially at rest, at the point A, with a horizontal displacement of 1.5 m from the line OC.

Given that the initial horizontal acceleration of the child is $-7.5\,\text{m s}^{-2}$

(a) show that $\lambda = 5$

(2)

Using the model,

(b) find an expression for the horizontal displacement of the child at time t.

(7)

Given that, when $t = 4.5$, the child is vertically below O,

(c) evaluate the model explaining your reasoning.

(2)

Question 6 continued

a) $\overset{o}{x}(0) = 0$

$x(0) = 1.5$

$\ddot{x}(0) = -7.5$

$-7.5 + 0 + \lambda(1.5) = 0$

$\lambda = 5$ //.

b) $m^2 + 2m + 5 = 0$

$(m+1)^2 = -4$

$m + 1 = \pm 2i$

$m = -1 \pm 2i$

$x = \cancel{A}e^{-t}(A\cos 2t + B\sin 2t)$

$1.5 = e^0(A)$

$A = 1.5$

$\dot{x} = -e^{-t}(A\cos 2t + B\sin 2t) + e^{-t}(-2A\sin 2t + 2B\cos 2t)$

$0 = (-A + 2B)$

$B = 0.75$

$x = e^{-t}(1.5\cos 2t + 0.75\sin 2t)$

c) $x = e^{-4.5}(1.5\cos 9 + 0.75\sin 9) = -0.0117...$

this is close to, so the model is quite acurate (good model).

7.

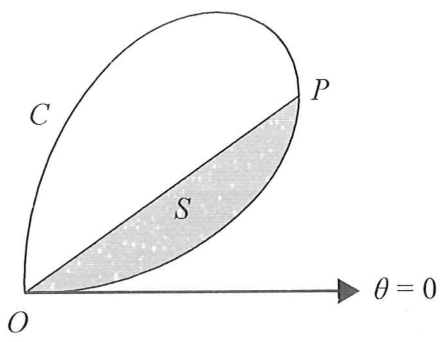


Figure 1

Figure 1 shows a sketch of curve C with polar equation

$$r = 3\sin 2\theta \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The point P on C has polar coordinates (R, ϕ). The tangent to C at P is perpendicular to the initial line.

(a) Show that $\tan\phi = \dfrac{1}{\sqrt{2}}$

(4)

(b) Determine the exact value of R.

(2)

The region S, shown shaded in Figure 1, is bounded by C and the line OP, where O is the pole.

(c) Use calculus to show that the exact area of S is

$$p\arctan\frac{1}{\sqrt{2}} + q\sqrt{2}$$

where p and q are constants to be determined.

Solutions relying entirely on calculator technology are not acceptable.

(7)

a) $\dfrac{dx}{d\theta} = 0 \qquad x = 3\cos\theta\sin 2\theta$

$\dfrac{dx}{d\theta} = \cancel{-3\sin\theta\sin 2\theta + 3\cos\theta(2\sin\theta\cos\theta)}$

$= -3\sin\theta\sin 2\theta + 3\cos\theta(2\cos 2\theta) = 0$

$\cancel{2\cos\theta(2\sin 2\theta + \cos^2\theta)} = 0$

$-3s(2cs) + 3c(1-2c^2) = 0$

$\cancel{c[2s^2+1-2c^2]=0}$

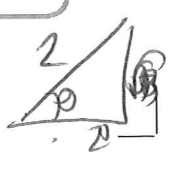
$\cancel{2c(1-c^2)+c(1-2c^2)=0}$

$\cancel{c[2-2c^2+1-2c^2]=0}$

$\cancel{c[3-4c^2]=0}$

$c^2 = \dfrac{3}{4}$

$\cos\theta = \dfrac{\sqrt{3}}{2}$



Question 7 continued

$x = 3\sin 2\theta \cos\theta$

$\frac{dx}{d\theta} = 6\cos 2\theta \cos\theta - 3\sin 2\theta \sin\theta = 0$

$\frac{1}{2}\tan 2\theta \tan\theta = 1$

$\tan 2\theta \tan\theta = 2$

$\frac{2t^2}{1-t^2} = 2$

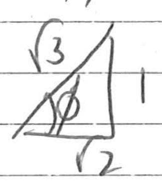
$2t^2 = 2 - 2t^2$

$4t^2 = 2$

$t^2 = \frac{1}{2}$

$\tan\phi = \frac{1}{\sqrt{2}}$ //

$r = 6\sin\theta\cos\theta$



$R = 6\sin\phi\cos\phi$

$= 6\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{6\sqrt{2}}{3} = 2\sqrt{2}$ //

c) $\frac{1}{2}\int_0^{\phi}(3\sin 2\theta)^2\, d\theta$

$\frac{9}{2}\int_0^{\phi}\sin^2 2\theta\, d\theta$

$\frac{9}{4}\int_0^{\phi} 1 - \cos 4\theta\, d\theta = \frac{9}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\phi}$

$\frac{9}{4}\left(\phi - \frac{\sin 4\phi}{4}\right)$

$\frac{9}{4}\left(\phi - \frac{\cancel{2}\sin 2\phi\cos 2\phi}{\cancel{4}\,2}\right)$

$\frac{9}{4}\left(\phi - \frac{\cancel{2}\sin\phi\cos\phi\,(1 - 2\sin^2\phi)}{\cancel{2}}\right)$

$\frac{9}{4}\left(\arctan\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{\sqrt{3}}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(1 - 2\left(\frac{1}{3}\right)\right)\right)$

$\frac{9}{4}\left(\arctan\left(\frac{1}{\sqrt{2}}\right) - \frac{\sqrt{2}}{9}\right)$

$\frac{9}{4}\arctan\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{4}$ //