STEP 1 Mathematics 2019 Question 11 Probability

STEP 1 Mathematics 2019
Question 11
Probability/ Game Theory

11 **(i)** Two people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p+q=1$). In each round of the procedure, both people toss their coins once. If both coins show heads, then both people go to the hotel; if both coins show tails, then both people go to the tea shop; otherwise, they continue to the next round. This process is repeated until a decision is made.

Show that the probability that they make a decision on the nth round is

$$(q^2+p^2)(2qp)^{n-1}\,.$$

Show also that the probability that they make a decision on or before the nth round is at least

$$1-\frac{1}{2^n}$$

whatever the value of p.

(ii) Three people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p+q=1$). In the first round of the procedure, all three people toss their coins once. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they continue to the next round.

In the next round the two people whose coins showed the same face toss again, but the third person just turns over his or her coin. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they go to the third round.

Show that the probability that they make a decision on or before the second round is at least $\frac{7}{16}$, whatever the value of p.

11 (i) Two people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p+q=1$). In each round of the procedure, both people toss their coins once. If both coins show heads, then both people go to the hotel; if both coins show tails, then both people go to the tea shop; otherwise, they continue to the next round. This process is repeated until a decision is made.

Show that the probability that they make a decision on the nth round is

$$(q^2+p^2)(2qp)^{n-1}\,.$$

Show also that the probability that they make a decision on or before the nth round is at least

$$1-\frac{1}{2^n}$$

whatever the value of p.

1st round.		2nd	3rd
pp	⇒ hotel	pp	pp
qq	⇒ tea shop	qq	qq
pq	⇒ repeat.	pq	pq
qp		qp	qp

2nd

$(pq+qp)\times(pp+qq)$
$=(2pq)\times(p^2+q^2)$

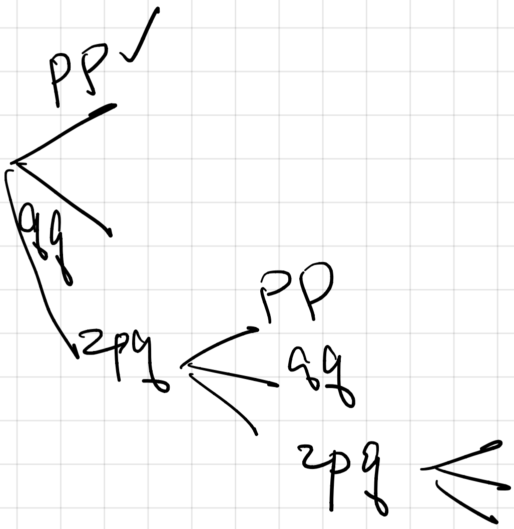
3rd

$(2pq)^2\times(p^2+q^2)$

4th

$(2pq)^3(p^2+q^2)$

nth round



11 **(i)** Two people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p+q=1$). In each round of the procedure, both people toss their coins once. If both coins show heads, then both people go to the hotel; if both coins show tails, then both people go to the tea shop; otherwise, they continue to the next round. This process is repeated until a decision is made.

Show that the probability that they make a decision on the nth round is

$$(q^2+p^2)(2qp)^{n-1}\,.$$

✓

Show also that the probability that they make a decision on or before the nth round is at least

$$1-\frac{1}{2^n}$$

whatever the value of p.

$$(2pq)^{n-1}(p^2+q^2) = (q^2+p^2)(2qp)^{n-1}$$

Attempt 1.

1st + 2nd + 3rd + ... nth

$$\underbrace{(p^2+q^2)}_{\text{1st}} + \underbrace{(p^2+q^2)(2pq)}_{\text{2nd}} + \underbrace{(p^2+q^2)(2pq)^2}_{\text{3rd}}$$

$$+ \cdots + \underbrace{(p^2+q^2)(2pq)^{n-1}}_{\text{nth round.}}$$

$$(p^2+q^2)\left(1+2pq+(2pq)^2+\ldots(2pq)^{n-1}\right)$$

$p+q=1$

$$S=\frac{a}{1-r}$$

$0<p<1$

$0<q<1$

$$(p^2+q^2)\left[\frac{1}{1-2pq}\right]$$

$$\left[(p+q)^2-2pq\right]\left[\frac{1}{1-2pq}\right]$$

$$\left[1-2pq\right]\left[\frac{1}{1-2pq}\right]$$

$$=1 \quad //$$

11 (i) Two people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p+q=1$). In each round of the procedure, both people toss their coins once. If both coins show heads, then both people go to the hotel; if both coins show tails, then both people go to the tea shop; otherwise, they continue to the next round. This process is repeated until a decision is made.

Show that the probability that they make a decision on the nth round is

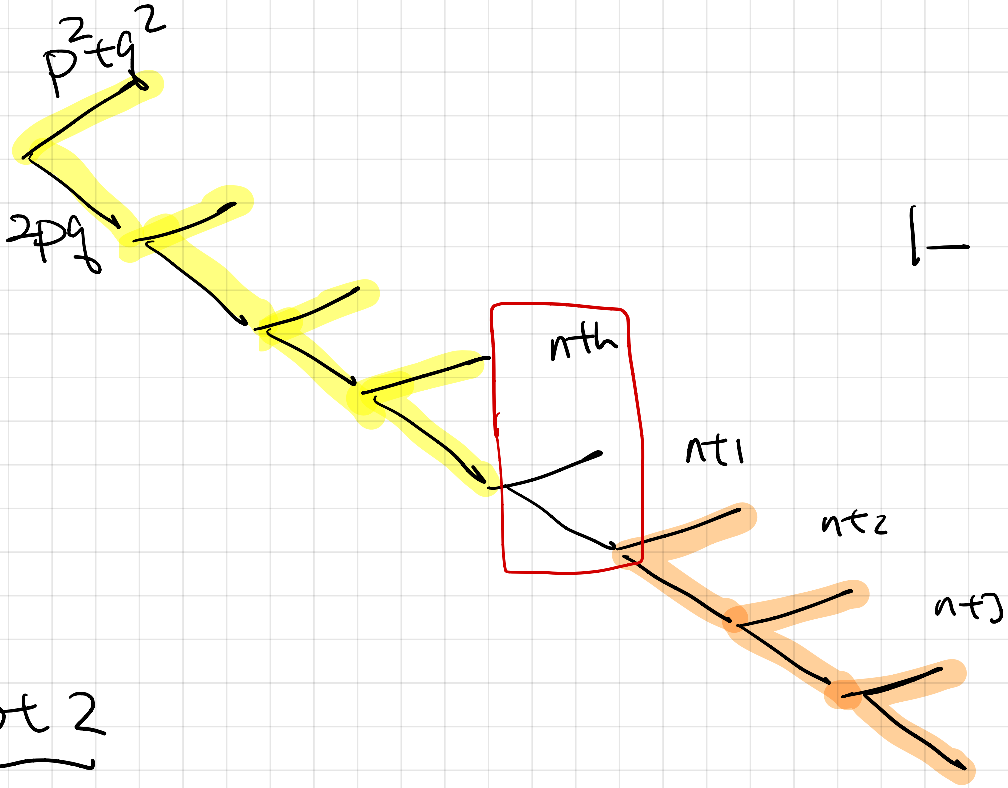
$$(q^2+p^2)(2qp)^{n-1}.$$

Show also that the probability that they make a decision on or before the nth round is at least

$$1-\frac{1}{2^n}$$

whatever the value of p.

$n+1 \quad n+2 \quad n+3 \ldots$



$1-$

$2pq < 1$

$r < 1$

Attempt 2

$$1-(2pq)^{n+1-1}(p^2+q^2)-(2pq)^{n+1}(p^2+q^2)$$

$$- \dots\dots\dots$$

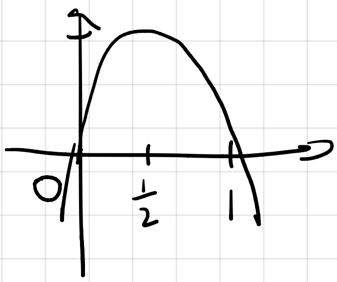
$$1-\left((2pq)^n(p^2+q^2)\left[1+2pq+(2pq)^2+\dots\right]\right)$$

$$D. = 1-(2pq)^n \quad //$$

$$D. = 1-2^n(pq)^n$$

$$p+q=1$$

$$2pq = 2p(1-p)$$



$$p-p^2$$

$$p=0 \quad p=1$$

$D = 1-(2pq)^n < 1-\left(\frac{2}{4}\right)^n$

$< 1-\left(\frac{1}{2}\right)^n$

$< 1-\frac{1}{2^n}$

$pq < \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$pq < \left(\frac{1}{4}\right)$

11 (i) Two people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p+q=1$). In each round of the procedure, both people toss their coins once. If both coins show heads, then both people go to the hotel; if both coins show tails, then both people go to the tea shop; otherwise, they continue to the next round. This process is repeated until a decision is made.

Show that the probability that they make a decision on the nth round is

$$(q^2+p^2)(2qp)^{n-1}.$$

Show also that the probability that they make a decision on or before the nth round is at least

$$1-\frac{1}{2^n}$$

whatever the value of p.

$(a+b+c)^2$ $(p+q)^2$ $(p+q)^3$

(ii) Three people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p + q = 1$). In the first round of the procedure, all three people toss their coins once. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they continue to the next round.

In the next round the two people whose coins showed the same face toss again, but the third person just turns over his or her coin. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they go to the third round.

Show that the probability that they make a decision on or before the second round is at least $\frac{7}{16}$, whatever the value of p.

ppp ✓ → hotel

qqq ✓ → tea shop.

1 → p^3+q^3

→ $3pq^2+3p^2q$

pqq $\quad$ ppq

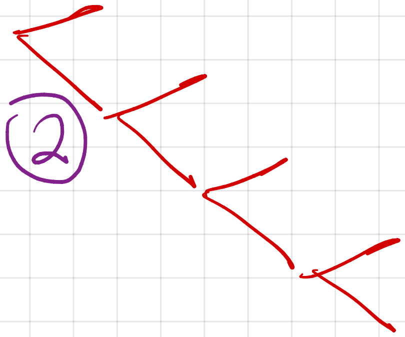
qpq $\quad$ pqp

qqp $\quad$ qpp

$\downarrow$ $\quad$ $\downarrow$

q^2 $\quad$ p^2

2



$$D=(p^3+q^3)+(3p\underset{q^2}{q^2}+3\underset{p^2}{p^2}q)$$

1st + 2nd

$$D= p^3+q^3+3pq^4+3p^4q$$

$\boxed{p+q=1}$

(ii) Three people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p + q = 1$). In the first round of the procedure, all three people toss their coins once. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they continue to the next round.

In the next round the two people whose coins showed the same face toss again, but the third person just turns over his or her coin. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they go to the third round.

Show that the probability that they make a decision on or before the second round is at least $\frac{7}{16}$, whatever the value of p.

$$D = p^3 + (1-p)^3 + 3p(1-p)^4 + 3p^4(1-p)$$

$$D = \cancel{p^3} + \cancel{-p^3} + 3p^2 - \cancel{3p} + 1$$
$$+ 3p(\cancel{p^4} - 4p^3 + 6p^2 - 4p + \cancel{1})$$
$$+ 3p^4 - \cancel{3p^5}$$

$$\frac{2 \times \overset{18}{3}}{54}$$

$$D = \cancel{3p^5} - 9p^4 + 18p^3 - 9p^2 + 1$$

$$\frac{dD}{dp} = -36p^3 + 54p^2 - 18p = 0$$

$$\frac{dD}{dp} = -4p^3 + 6p^2 - 2p = 0$$

$$\frac{dD}{dp} = -2p^3 + 3p^2 - p = 0$$

$p=0$ OR $-2p^2+3p-1=0$

(rej) $2p^2-3p+1=0$

$\frac{dD}{dp}=0$ $p(2p-1)(p-1)=0$

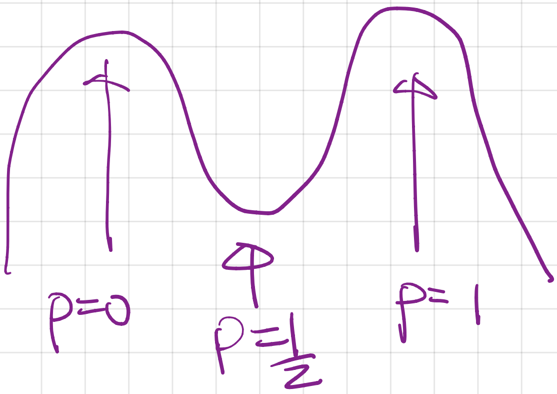
$p=0$ | $\boxed{p=\frac{1}{2}}$ | $p=1$
(rej) | | (rej)
$q=1$ | $q=\frac{1}{2}$ | $q=0$

$D= \cancel{3p^5} - 9p^4 + 18p^3 - 9p^2 + 1$

$D= -9\left(\frac{1}{16}\right) + 18\frac{1}{8} - \frac{9}{4} + 1$

$D= \frac{-9}{16} + \cancel{\frac{36}{16}} - \cancel{\frac{36}{16}} + \frac{16}{16}$

$D= \frac{7}{16}$



(ii) Three people adopt the following procedure for deciding where to go for a cup of tea: either to a hotel or to a tea shop. Each person has a coin which has a probability p of showing heads and q of showing tails (where $p + q = 1$). In the first round of the procedure, all three people toss their coins once. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they continue to the next round.

In the next round the two people whose coins showed the same face toss again, but the third person just turns over his or her coin. If all three coins show heads, then all three people go to the hotel; if all three coins show tails, then all three people go to the tea shop; otherwise, they go to the third round.

Show that the probability that they make a decision on or before the second round is at least $\frac{7}{16}$, whatever the value of p.

min of P attains at $p = \frac{1}{2}$

is $P = \frac{7}{16}$, $p = \frac{1}{2}$ //