Pearson Edexcel Level 3 GCE

May–June 2022 Assessment Window

Syllabus reference **9FM0**

Further Mathematics

Advanced
Advance Information

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Mon 6th June 2022 Further Maths Revision Session 2

Paper 9FM0/02 Further Mathematics Core Pure 2

- Proof by induction; Use matrices to represent linear transformations in 2-D
- Complex numbers: Multiplication and division
- Complex numbers; Addition and subtraction; simple loci in the Argand diagram
- Matrices: Solution of three simultaneous equations
- The relationship between roots and coefficients of polynomial equations
- Differentiate inverse trigonometric functions
- Vectors; Equation of a straight line, scalar product, perpendicular distance from a point to a plane
- Polar coordinates: Area enclosed by a curve, tangents
- Differentiation of hyperbolic functions; Maclaurin series

ACH: Table of content:
8 Questions based on the topics in Advanced information 2022
Practice Paper for Core Pure Paper 2.

	Marks	Source
Paper 9FM0/02 Further Mathematics Core Pure 2		
• Proof by induction; Use matrices to represent linear transformations in 2-D	13	CIE FP1 Jun 2020
• Complex numbers: Multiplication and division	9	CIE P3 March 2021
• Complex numbers; Addition and subtraction; simple loci in the Argand diagram		
• Matrices: Solution of three simultaneous equations	6	CIE FP1 June 2015
• The relationship between roots and coefficients of polynomial equations	4	CIE FP1 June 2016
• Differentiate inverse trigonometric functions	9	CIE FP2 Oct 2020
• Vectors; Equation of a straight line, scalar product, perpendicular distance from a point to a plane	11	CIE FP1 Oct 2020
• Polar coordinates: Area enclosed by a curve, tangents	13	Edexcel IAL F2 Oct 2020
• Differentiation of hyperbolic functions; Maclaurin series	10	OCR FP2 June 2013
Total	75	
• Matrices: Solution of three simultaneous equations	8	CIE FP1 Oct 2014
• The relationship between roots and coefficients of polynomial equations	6	CIE FP1 June 2020

- Proof by induction; Use matrices to represent linear transformations in 2-D

6 Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

13 Marks, 15.6 mins

13 Marks, 15.6 mins

(a)

$A\Delta = \Delta'$

$A^{-1}A\Delta = A^{-1}\Delta'$

$\Delta = A^{-1}\Delta'$

$\det A = 2$

$2(d) = 30$

$d = 15$

$2 \cdot \Delta = \Delta'$

$\Delta = \frac{1}{2}\Delta'$

(b)

Basis $A^1 = \begin{pmatrix} 2 & 0 \\ 2-1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

Assume tre for $n=k$

$A^k = \begin{pmatrix} 2^k & 0 \\ 2^k-1 & 1 \end{pmatrix}$

$A^{k+1} = \begin{pmatrix} 2^k & 0 \\ 2^k-1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2^k \cdot 2 & 0 \\ 2(2^k-1)+1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1}-1 & 1 \end{pmatrix}$

tre for $n=k+1$

$\therefore$ proved tre for $n=1$

Assumed tre for $n=k$

$\Rightarrow$ tre by MI.

$\begin{pmatrix} 2^n & 0 \\ 2^n-1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{pmatrix}$

$\begin{pmatrix} 2^n & 0 \\ 2^n+32 & 0 \end{pmatrix}\begin{pmatrix} x \\ 2x \end{pmatrix} = \begin{pmatrix} 2^n x \\ (2^n+32)x \end{pmatrix}$

$2 \cdot 2^n x = 2^n x + 32x$

$2^{n+1} = 2^n + 32$

$2^n = 32$

$n = 5$

Question	Answer	Marks
6(a)	$\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1} = \frac{1}{2}$	M1 A1
	$d = 15$	A1
		3
6(b)	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 0 \\ 2^1 - 1 & 1 \end{pmatrix}$ so true when $n = 1$.	B1
	Assume that it is true for $n = k$, so $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}$.	B1
	Then $\mathbf{A}^{k+1} = \begin{pmatrix} 2^k & 0 \\ 2^k - 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2(2^k - 1) + 1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 \\ 2^{k+1} - 1 & 1 \end{pmatrix}$	M1A1
	So, it is also true for $n = k + 1$. Hence, by induction, true for all positive integers.	A1
		5
6(c)	$\mathbf{A}^n\mathbf{B} = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}$	M1A1
	$\begin{pmatrix} 2^n & 0 \\ 2^n + 32 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2^n x \\ (2^n + 32)x \end{pmatrix}$	B1
	$(2^n + 32)x = 2^{n+1}x \Rightarrow 2^n = 32 \Rightarrow n = 5$	M1 A1
		5

- Complex numbers: Multiplication and division
- Complex numbers; Addition and subtraction; simple loci in the Argand diagram

8 The complex numbers u and v are defined by $u = -4 + 2\mathrm{i}$ and $v = 3 + \mathrm{i}$.

(a) Find $\frac{u}{v}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(b) Hence express $\frac{u}{v}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where r and θ are exact. [2]

In an Argand diagram, with origin O, the points A, B and C represent the complex numbers u, v and $2u + v$ respectively.

(c) State fully the geometrical relationship between OA and BC. [2]

(d) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

9 marks, 10.8 mins

a)

$u = -4+2i$

$v = 3+i$

$$\frac{u}{v} = \frac{-4+2i}{3+i} \cdot \frac{3-i}{3-i}$$

$$= \frac{-12+6i+4i+2}{9+1}$$

$$= \frac{-10+10i}{10} = -1+i$$

b)

$\sqrt{2}\, e^{\frac{3\pi}{4} i}$

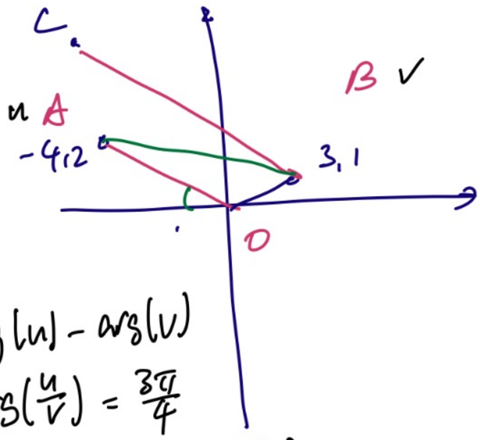
c)

$2(-4+2i)+3+i$

$= -8+3+4i+i$

$= -5+5i$

OA // BC



$\arg(u) - \arg(v)$

$= \arg\left(\frac{u}{v}\right) = \frac{3\pi}{4}$

(answer in part b)

Question	Answer	Marks	Guidance
8(a)	Multiply numerator and denominator by 3 – i	**M1**	OE
	Obtain numerator – 10 + 10i or denominator 10	**A1**	
	Obtain final answer – 1 + i	**A1**	
		3	
8(b)	State or imply $r = \sqrt{2}$	**B1 FT**	
	State or imply that $\theta = \frac{3}{4}\pi$	**B1 FT**	
		2	
8(c)	State that *OA* and *BC* are parallel	**B1**	
	State that $BC = 2OA$	**B1**	
		2	

- ## Matrices: Solution of three simultaneous equations

2 Find the value of the constant k for which the system of equations

$$\begin{aligned} 2x - 3y + 4z &= 1, \\ 3x - y \qquad &= 2, \\ x + 2y + kz &= 1, \end{aligned}$$

does not have a unique solution. [2]

For this value of k, solve the system of equations. [4]

6 marks, 7.2 mins

$$M\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$M=\begin{pmatrix}2&-3&4\\3&-1&0\\1&2&k\end{pmatrix}$$

6 marks, 7.2 mins

$$\det M = 2\begin{bmatrix}-1&0\\2&k\end{bmatrix}-(-3)\begin{bmatrix}3&0\\1&k\end{bmatrix}+4\begin{bmatrix}3&-1\\1&2\end{bmatrix}$$

$$=2(-k)+3(3k)+4(6+1)$$

$$=-2k+9k+28$$

$\det M=0$

$7k=-28$

$k=-4$

$$\begin{pmatrix}2&-3&4\\3&-1&0\\1&2&-4\end{pmatrix}$$

$2x-3y+4z=1$

$3x-y+0=2$

$x+2y-4z=1$

$z=0$

$2x-3y=1$

$x+2y=1$

$x=\frac{5}{7}$ $y=\frac{1}{7}$ $z=0$

check

$3x-y=2$

$2x-3y=1$

$$\begin{pmatrix}2\\-3\\4\end{pmatrix}\times\begin{pmatrix}3\\-1\\5\end{pmatrix}=\begin{pmatrix}4\\12\\7\end{pmatrix}$$

$$r=\begin{pmatrix}\frac{5}{7}\\\frac{1}{7}\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\12\\7\end{pmatrix}$$

$x=t$

$y=3t-2$

$z=\frac{7}{4}t-\frac{5}{4}$

$$\begin{pmatrix}0\\-2\\-\frac{5}{4}\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\7/4\end{pmatrix}$$

$$\begin{pmatrix}0\\-2\\-\frac{5}{4}\end{pmatrix}+\lambda\begin{pmatrix}4\\12\\7\end{pmatrix}$$

$2x+4z=1$

$3x=2$

$x-4z=1$

$x=\frac{2}{3}$ $y=0$ $z=\frac{-1}{12}$

2	$\begin{vmatrix} 2 & -3 & 4 \\ 3 & -1 & 0 \\ 1 & 2 & k \end{vmatrix} = 0 \Rightarrow 7k = -28 \Rightarrow k = -4$	M1A1 (2)
	Add 1st and 3rd $\Rightarrow 3x - y = 2$ (Same as 2nd) (OE) Set $x = t$ (for example) (OE) $\Rightarrow y = 3t - 2,\ z = \frac{7}{4}t - \frac{5}{4}$ (OE) – many forms	M1 M1 A1A1 (4) **Total: 6**

ACH: go over alternative methods, eg,
1) set x=0, find a point, then set y=0 find another point and find equation of straight line
2) the direction vector can be found by using cross product of the normals, since the direction is perpendicular to all the normals.

1 The roots of the cubic equation $2x^3 + x^2 - 7 = 0$ are α, β and γ. Using the substitution $y = 1 + \frac{1}{x}$, or otherwise, find the cubic equation whose roots are $1 + \frac{1}{\alpha}$, $1 + \frac{1}{\beta}$ and $1 + \frac{1}{\gamma}$, giving your answer in the form $ay^3 + by^2 + cy + d = 0$, where a, b, c and d are constants to be found. [4]

4 marks, 4.8 mins

$y = 1 + \frac{1}{x}$

$y - 1 = \frac{1}{x}$

$x = \frac{1}{y-1}$

4 marks, 4.8 mins

$2\left(\frac{1}{y-1}\right)^3 + \left(\frac{1}{y-1}\right)^2 - 7 = 0$

$\frac{2}{(y-1)^3} + \frac{1}{(y-1)^2} = 7$

$2 + (y-1) = 7(y-1)^3$

$2 + (y-1) = 7(y^3 - 3y^2 + 3y - 1)$

$0 = 7y^3 - 21y^2 + 21y - 7 - 2 - y + 1$

$0 = 7y^3 - 21y^2 + 20y - 8$

Qu	Solution	Part Marks
1	$y = 1 + \frac{1}{x} \Rightarrow x = \frac{1}{y-1}$	M1
	$\frac{2}{(y-1)^3} + \frac{1}{(y-1)^2} - 7 = 0 \Rightarrow 2 + (y-1) - 7(y-1)^3 = 0$	A1
	$\Rightarrow 7(y^3 - 3y^2 + 3y - 1) - y + 1 - 2 = 0 \Rightarrow 7y^3 - 21y^2 + 20y - 8 = 0$	M1A1 [4]
	ALT METHOD: ∑α, ∑αβ, αβγM1 A1, ∑(1+1/α) etc M1 A1√	

- **Differentiate inverse trigonometric functions**

5 It is given that

$$x = \sinh^{-1} t, \quad y = \cos^{-1} t,$$

where $-1 < t < 1$.

(a) By differentiating $\cos y$ with respect to t, show that $\dfrac{dy}{dt} = -\dfrac{1}{\sqrt{1-t^2}}$. [4]

(b) Find $\dfrac{d^2y}{dx^2}$ in terms of t, simplifying your answer. [5]

9 marks, 10.8 mins

$y = \arccos t$

$\cos y = t$

$-\sin y \frac{dy}{dt} = 1$

$\frac{dy}{dt} = \frac{1}{-\sin y}$

$\frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

$\sin^2 y + \cos^2 y = 1$

$\sin y = \sqrt{1-t^2}$

9 marks, 10.8 mins

$\sinh x = t$

$\cosh x \frac{dx}{dt} = 1$

$\frac{dx}{dt} = \frac{1}{\sqrt{1+t^2}}$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh x = \sqrt{1+t^2}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-t^2}} \cdot \frac{\sqrt{1+t^2}}{1}$

$\frac{d}{dt} \cdot \frac{dt}{dx} \quad -\sqrt{\frac{1+t^2}{1-t^2}} = \frac{d^2y}{dx^2}$

$\Rightarrow \frac{d}{dt}\frac{dy}{dx} = -\frac{(1-t^2)^{\frac{1}{2}}(\frac{1}{2})(2t)(1+t^2)^{-\frac{1}{2}} - (1+t^2)^{\frac{1}{2}}(\frac{1}{2})(-2t)(1-t^2)^{-\frac{1}{2}}}{1-t^2}$

$= \frac{t(1-t^2)^{\frac{1}{2}}(1+t^2)^{-\frac{1}{2}} + (1+t^2)^{\frac{1}{2}}(t)(1-t^2)^{-\frac{1}{2}}}{1-t^2}$

$= t(1-t^2)^{-\frac{1}{2}}(1+t^2)^{-\frac{1}{2}} + (1+t^2)^{\frac{1}{2}}(t)(1-t^2)^{-3/2}$

$= (1-t^2)^{-3/2}(t)\left[(1-t^2)^1 + (1+t^2)^1\right]$
$(1+t^2)^{-1/2}$

$= -2t(1-t^2)^{-3/2}(1+t^2)^{-\frac{1}{2}}$

$\frac{d^2y}{dx^2} = -2t(1-t^2)^{-3/2}(1+t^2)^{-\frac{1}{2}} \cdot (1+t^2)^{\frac{1}{2}}$

$= -2t(1-t^2)^{-3/2}$

Question	Answer	Marks	Guidance
5(a)	$-\sin y\dfrac{dy}{dt}=1$	M1 A1	Differentiates both sides with respect to t.
	$0<y<\pi \Rightarrow \sin y>0 \Rightarrow -\sqrt{1-\cos^2 y}\dfrac{dy}{dt}=1$	M1	Applies $\sin^2 y+\cos^2 y=1$.
	$\dfrac{dy}{dt}=-\dfrac{1}{\sqrt{1-t^2}}$	A1	AG, justifies taking positive square root.
		4	
5(b)	$\dfrac{dx}{dt}=\dfrac{1}{\sqrt{1+t^2}}$	B1	
	$\dfrac{dy}{dx}=-\sqrt{\dfrac{1+t^2}{1-t^2}}$	B1	Finds first derivative.
	$\dfrac{d}{dt}\left(-\sqrt{\dfrac{1+t^2}{1-t^2}}\right)=-\dfrac{t\left(1-t^2\right)^{\frac{1}{2}}\left(1+t^2\right)^{-\frac{1}{2}}+t(1+t^2)^{\frac{1}{2}}\left(1-t^2\right)^{-\frac{1}{2}}}{1-t^2}$	M1	Differentiates $\dfrac{dy}{dx}$ with respect to t.
	$\dfrac{d^2y}{dx^2}=\dfrac{d}{dt}\left(-\sqrt{\dfrac{1+t^2}{1-t^2}}\right)\times\dfrac{dt}{dx}$	M1	Applies chain rule.
	$=-\dfrac{t\left(\left(1-t^2\right)^{\frac{1}{2}}+(1+t^2)\left(1-t^2\right)^{-\frac{1}{2}}\right)}{1-t^2}$ $\left(=-\dfrac{2t}{\left(1-t^2\right)^{\frac{3}{2}}}\right)$	A1	OE (simplified).
		5	

- ## Vectors; Equation of a straight line, scalar product, perpendicular distance from a point to a plane

4 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the planes OAB and ABC. [4]

11 marks, 13.2 mins

A

B C

O

② A: $\begin{pmatrix}-1\\1\\2\end{pmatrix}$ B: $\begin{pmatrix}-2\\-1\\0\end{pmatrix}$ C: $\begin{pmatrix}2\\0\\2\end{pmatrix}$

$\vec{AB} = \begin{pmatrix}-2\\-1\\0\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$

$\vec{AC} = \begin{pmatrix}2\\0\\2\end{pmatrix} - \begin{pmatrix}-1\\1\\2\end{pmatrix} = \begin{pmatrix}3\\-1\\0\end{pmatrix}$

$\vec{AB} \times \vec{AC} = \begin{pmatrix}-2\\-6\\7\end{pmatrix}$

$a \cdot n = r \cdot n$

$\begin{pmatrix}-1\\1\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\-6\\7\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-2\\-6\\7\end{pmatrix}$

$2 - 6 + 14 = -2x - 6y + 7z$

$-2x - 6y + 7z = 10$

b) $\dfrac{|-2(0) - 6(0) + 7(0) - 10|}{\sqrt{2^2 + 6^2 + 7^2}}$

$= \dfrac{10}{\sqrt{89}}$

c)

OA × OB

$= \begin{pmatrix}-1\\1\\2\end{pmatrix} \times \begin{pmatrix}-2\\1\\0\end{pmatrix}$

$= \begin{pmatrix}2\\-4\\3\end{pmatrix}$

$\begin{pmatrix}2\\-4\\3\end{pmatrix} \cdot \begin{pmatrix}-2\\-6\\7\end{pmatrix} = \; \ldots\ldots$

$\alpha = 36.19$ //

Question	Answer	Marks	Guidance
4(a)	$\overrightarrow{AB} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ $\overrightarrow{AC} = 3\mathbf{i} - \mathbf{j}$	B1	Finds direction vectors of two lines in the plane. $\overrightarrow{BC} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 3 & -1 & 0 \end{vmatrix} = \begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix}$	M1 A1	Finds normal to the plane *ABC*.
	$-2(-1) - 6(1) + 7(2) = 10 \Rightarrow -2x - 6y + 7z = 10$	M1 A1	Substitutes point.
	Alternative method for question 4(a)		
	Setting up 3 equations using points given.	M1	
	$-2x - 6y + 7z = 10$	A1 A1 A1 A1	OE
		5	
4(b)	$\frac{10}{\sqrt{2^2 + 6^2 + 7^2}} = \frac{10}{\sqrt{89}}$ OE	M1 A1	Divides by magnitude of normal vector. 1.06…
		2	

Question	Answer	Marks	Guidance
4(c)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -2 & -1 & 0 \end{vmatrix} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}$	M1 A1	Finds normal to the plane *OAB*.
	$\begin{pmatrix} -2 \\ -6 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} = \sqrt{89}\sqrt{29}\cos\theta$	M1	Uses dot product correctly.
	36.2°	A1	
		4	

- # Polar coordinates: Area enclosed by a curve, tangents

Pearson Edexcel International
Advanced Level
Further Pure Mathematics F2
WFM02/01
Friday 09 October 2020

13 marks, 15.6 mins

7.

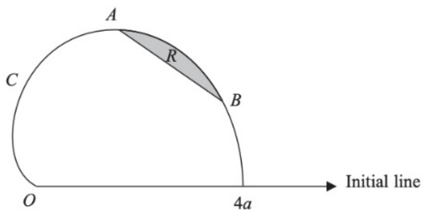


Figure 1

The curve C, shown in Figure 1, has polar equation

$$r = 2a(1 + \cos\theta) \qquad 0 \leqslant \theta \leqslant \pi$$

where a is a positive constant.

The tangent to C at the point A is parallel to the initial line.

(a) Determine the polar coordinates of A.

(6)

The point B on the curve has polar coordinates $\left(a\left(2+\sqrt{3}\right), \frac{\pi}{6}\right)$

The finite region R, shown shaded in Figure 1, is bounded by the curve C and the line AB.

(b) Use calculus to determine the exact area of the shaded region R.

Give your answer in the form

$$\frac{a^2}{4}\left(d\pi - e + f\sqrt{3}\right)$$

where d, e and f are integers.

(7)

• Polar coordinates: Area enclosed by a curve, tangents

7.

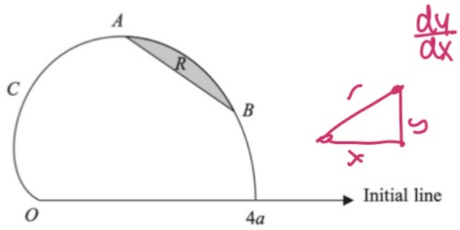


Figure 1

The curve C, shown in Figure 1, has polar equation

$$r = 2a(1 + \cos\theta) \qquad 0 \leqslant \theta \leqslant \pi$$

where a is a positive constant.

The tangent to C at the point A is parallel to the initial line.

(a) Determine the polar coordinates of A. (6)

$\frac{dy}{dx} = 0 \quad \frac{dy}{d\theta} = 0$

$\frac{y}{r} = \sin\theta$

$\frac{y}{\sin\theta} = 2a(1+\cos\theta)$

$y = 2a\sin\theta + 2a\sin\theta\cos\theta$

$y = 2a\sin\theta + a\sin 2\theta$

$\frac{dy}{d\theta} = 2a\cos\theta + 2a\cos 2\theta$

$\frac{dy}{d\theta} = 0$

$\cos\theta + \cos^2\theta - \sin^2\theta = 0$

$2\cos^2\theta + \cos\theta - 1 = 0$

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2} \quad \cos\theta = -1$

$\theta = \frac{\pi}{3} \quad r = 2a(1+\frac{1}{2})$

$r = 3a$

Area of $\triangle$ OAB

$= \frac{1}{2}(3a)(2a + a\sqrt{3})\sin(\frac{\pi}{6})$

$= \frac{3}{4}a^2(2+\sqrt{3})$

The point B on the curve has polar coordinates $\left(a(2+\sqrt{3}), \frac{\pi}{6}\right)$

The finite region R, shown shaded in Figure 1, is bounded by the curve C and the line AB.

(b) Use calculus to determine the exact area of the shaded region R.

Give your answer in the form

$$\frac{a^2}{4}\left(d\pi - e + f\sqrt{3}\right)$$

where d, e and f are integers. (7)

Area of sector $= \int \frac{1}{2}r^2\,d\theta = \frac{1}{2}(4a^2)\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1+\cos\theta)^2\,d\theta$

$= 2a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2\theta + 2\cos\theta + 1\,d\theta$

$= 2a^2\int \frac{\cos 2\theta + 1}{2} + 2\cos\theta + 1\,d\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \cos^2\theta - (1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$

$= 2a^2\left[\frac{\sin 2\theta}{4} + \frac{3}{2}\theta + 2\sin\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= 2a^2\left[\frac{\sqrt{3}}{8} + \frac{\pi}{2} + \sqrt{3} - \frac{\sqrt{3}}{8} - \frac{\pi}{4} - 1\right]$

$= 2a^2\left[\frac{\pi}{4} + \sqrt{3} - 1\right]$

Area $= a^2\left[\frac{2\pi}{4} + 2\sqrt{3} - 2 - \frac{6}{4} - \frac{3\sqrt{3}}{4}\right]$

$= \frac{a^2}{4}\left[2\pi - 14 + 5\sqrt{3}\right]$

Question Number	Scheme	Marks
7 (a)	$r\sin\theta = 2a\sin\theta + 2a\sin\theta\cos\theta$ OR $r\sin\theta = 2a\sin\theta + a\sin 2\theta$	B1
	$\frac{d(r\sin\theta)}{d\theta} = 2a\cos\theta + 2a\cos^2\theta - 2a\sin^2\theta$ OR $\frac{d(r\sin\theta)}{d\theta} = 2a\cos\theta + 2a\cos 2\theta$	M1 A1
	$2\cos^2\theta + \cos\theta - 1 = 0$ terms in any order	
	$(2\cos\theta - 1)(\cos\theta + 1) = 0$	
	$\cos\theta = \frac{1}{2}$ $\theta = \frac{\pi}{3}$ ($\theta = \pi$ need not be seen)	dM1A1
	$r = 2a \times \frac{3}{2} = 3a$	A1 (6)
(b)	Area $= \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4a^2(1+\cos\theta)^2 d\theta$	
	$= 2a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1 + 2\cos\theta + \cos^2\theta)d\theta$	M1
	$= 2a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(1 + 2\cos\theta + \frac{1}{2}(\cos 2\theta + 1)\right)d\theta$	M1
	$= 2a^2\left[\theta + 2\sin\theta + \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$	dM1A1
	$= 2a^2\left[\frac{\pi}{3} + \sqrt{3} + \frac{1}{4}\times\frac{\sqrt{3}}{2} + \frac{\pi}{6} - \left(\frac{\pi}{6} + 1 + \frac{1}{4}\times\frac{\sqrt{3}}{2} + \frac{\pi}{12}\right)\right]$	M1 NB: A1 on e-PEN
	$= 2a^2\left(\frac{\pi}{4} + \sqrt{3} - 1\right)$	
	Area of $\Delta OAB = \frac{1}{2}\times 3a \times (2+\sqrt{3})a \times \sin\frac{\pi}{6}$ $\left(= \frac{3}{4}a^2(2+\sqrt{3})\right)$	
	Shaded area $= 2a^2\left(\frac{\pi}{4} + \sqrt{3} - 1\right) - \frac{3}{4}a^2(2+\sqrt{3}) = \frac{a^2}{4}(2\pi - 14 + 5\sqrt{3})$	M1A1cao (7)
		[13]

Question Number	Scheme	Marks
(a)		
B1	Multiply r by $\sin\theta$ Award if not seen explicitly but a correct result following use of double angle formula is seen	
M1	Differentiate $r\sin\theta$ or $r\cos\theta$ (using product rule or using double angle formula first)	
A1	Correct derivative for $r\sin\theta$	
dM1	Use $\sin^2\theta + \cos^2\theta = 1$ to form a 3TQ in $\cos\theta$ and attempt its solution by a valid method	
A1	Correct value for θ	
A1	Correct r	
(b)		
M1	Use area $= \frac{1}{2}\int r^2 d\theta$ with $r = 2a + 2a\cos\theta$, no limits needed,	
M1	Use a double angle formula to obtain a function ready for integrating (Alt method uses integration by parts – may be seen)	
dM1	Attempt the integration $\cos 2\theta \to \frac{1}{k}\sin 2\theta$ $k = \pm 2$ or ± 1	
A1	Correct integration,	
M1	Substitute the limits (need not be simplified). Limits $\frac{\pi}{6}$ and their θ from (a) provided this is $> \frac{\pi}{6}$ NB: A1 on e-PEN	
M1	Obtain the area of ΔOAB and subtract from their previous area	
A1	Correct answer	

OCR
A2 GCE MATHEMATICS
4726/01 Further Pure Mathematics 2
Thursday 13 June 2013

- ## Differentiation of hyperbolic functions; Maclaurin series

3 It is given that $f(x) = \tanh^{-1}\left(\frac{1-x}{3+x}\right)$ for $x > -1$.

(i) Show that $f''(x) = \frac{1}{2(x+1)^2}$. **[6]**

(ii) Hence find the Maclaurin series for f(x) up to and including the term in x^2. **[4]**

10 marks, 12 mins

$y = \tanh^{-1}\left(\frac{1-x}{3+x}\right)$

$\tanh y = \frac{1-x}{3+x}$

$\frac{dy}{dx}\operatorname{sech}^2 y = \frac{(3+x)(-1)-(1-x)(1)}{(3+x)^2}$

$\frac{dy}{dx} = \frac{-3-x-1+x}{(3+x)^2} \cdot \frac{1}{1-\tanh^2 x}$

$= \frac{-4}{(3+x)^2} \cdot \frac{1}{1-\frac{(1-x)^2}{(3+x)^2}}$

$= \frac{-4}{(3+x)^2} \cdot \frac{(3+x)^2}{(3+x)^2-(1-x)^2}$

$= \frac{-4}{(3+x)^2-(1-x)^2} = \frac{-4}{(3+x+1-x)(3+x-1+x)}$

$= \frac{-1}{2x+2}$

$= \frac{-1}{2(x+1)}$

$-\sinh^2 x + \cosh^2 x = 1$

$-\tanh^2 x + 1 = \operatorname{sech}^2 x$

$\frac{d^2y}{dx^2} = \frac{-1}{2} \cdot \frac{0-(1)(1)}{(x+1)^2}$

$= \frac{1}{2} \cdot \frac{1}{(x+1)^2}$

$x=0$

$\tanh y = \frac{1}{3}$

$\frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{1}{3}$

$3(e^y)^2 - 3 = (e^y)^2 + 1$

$2(e^y)^2 = 4$

$e^y = \pm\sqrt{2}$

$y = \ln\sqrt{2}$

$x=0 \quad \frac{dy}{dx} = \frac{-1}{2}$

$x=0 \quad \frac{d^2y}{dx^2} = \frac{1}{2}$

$y = \ln\sqrt{2} - \frac{1}{2}x + \frac{1}{4}x^2 + \dots$

3	(i)		$\frac{dy}{dx} = \frac{1}{1-\left(\frac{1-x}{3+x}\right)^2} \times \frac{-(3+x)-(1-x)}{(3+x)^2}$	B1 M1 A1	Sight of standard diffn for $\tanh^{-1}x$ Fn of fn **and** diffn of quotient Soi correct quotient (i.e. correct expression for 2nd part)	
			$\Rightarrow \frac{dy}{dx} = \left(\frac{-4}{(3+x)^2-(1-x)^2}\right) = \frac{k}{1+x}$	A1		
			$\Rightarrow \frac{dy}{dx} = \frac{-1}{2(1+x)}$	A1	Correct for y'	
			$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2(1+x)^2}$	A1 **[6]**	2nd diffn (NB AG)	

Question			Answer	Marks	Guidance	
3	(ii)		When $x = 0, y = \tanh^{-1}\frac{1}{3}$ or $\frac{1}{2}\ln 2$ or $\ln\sqrt{2}$	B1	For 1st value (needs to be exact)	
			$\frac{dy}{dx} = -\frac{1}{2}$ $\frac{d^2y}{dx^2} = \frac{1}{2}$	B1	For both	
			$\Rightarrow y = \tanh^{-1}\frac{1}{3} + \left(-\frac{1}{2}\right)x + \left(\frac{1}{2}\right)\frac{x^2}{2}$	M1	Use of correct Maclaurin's series	
			$= \tanh^{-1}\frac{1}{3} - \frac{1}{2}x + \frac{x^2}{4}$	A1 **[4]**	Accept 0.347	

Bonus
Content

• Matrices: Solution of three simultaneous equations

5 Find the value of a for which the system of equations

$$\begin{aligned} x - y + 2z &= 4, \\ x + ay - 3z &= b, \\ x - y + 7z &= 13, \end{aligned}$$

where a and b are constants, has no unique solution. [3]

Taking a as the value just found,

(i) find the general solution in the case $b = -5$, [4]

(ii) interpret the situation geometrically in the case $b \neq -5$. [1]

8 marks, 9.6 mins

5	$\begin{vmatrix} 1 & -1 & 2 \\ 1 & -1 & -3 \\ 1 & -1 & 7 \end{vmatrix} = 0$	M1
	$\Rightarrow 5a + 5 = 0 \Rightarrow a = -1$	A1A1 (3)
	$\begin{pmatrix} 1 & -1 & 2 & \vert & 4 \\ 1 & -1 & -3 & \vert & -5 \\ 1 & -1 & 7 & \vert & 13 \end{pmatrix} \rightarrow \ldots \rightarrow \begin{pmatrix} 1 & -1 & 2 & \vert & 4 \\ 0 & 0 & 5 & \vert & 9 \\ 0 & 0 & 0 & \vert & 0 \end{pmatrix}$, or by elimination methods,	M1
	$x - y + 2z = 4$ $5z = 9$	A1
	$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0 \\ 1.8 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (OE)	M1A1 (4)
	Planes form a prism, or words to that effect.	B1 (1) **[8]**

Bonus Content

- ## The relationship between roots and coefficients of polynomial equations

2 The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

6 marks, 7.2 mins

Question	Answer	Marks
2(a)	$y = x^2$	B1
	$6y^{\frac{3}{2}} + py - 3y^{\frac{1}{2}} - 5 = 0 \Rightarrow y^{\frac{1}{2}}(6y-3) = -py+5$ $y(6y-3)^2 = (-py+5)^2 \Rightarrow y(36y^2 - 36y + 9) = p^2y^2 - 10py + 25$	M1
	$36y^3 - (p^2+36)y^2 + (10p+9)y - 25 = 0$	A1
		3
2(b)(i)	$\alpha^2 + \beta^2 + \gamma^2 = \frac{p^2+36}{36}$	B1
	$\frac{p^2+36}{36} = -\frac{2p}{6} \Rightarrow p^2 + 12p + 36 = 0$	M1
	$p = -6$	A1
		3
2(b)(ii)	$6(\alpha^3 + \beta^3 + \gamma^3) = 6(\alpha^2 + \beta^2 + \gamma^2) + 3(\alpha + \beta + \gamma) + 15$	M1
	$\alpha^3 + \beta^3 + \gamma^3 = 5$	A1
		2