1 *O*, *A* and *B* are fixed points such that

$$\overrightarrow{OA} = 4\mathbf{i} + 3\mathbf{j} \qquad \overrightarrow{OB} = 8\mathbf{i} + p\mathbf{j} \qquad \text{and} \qquad \left|\overrightarrow{AB}\right| = 2\sqrt{13}$$

(a) Find the possible values of *p*.

(3)

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= -\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 8 \\ p \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ p-3 \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = 2\sqrt{13}$$

$$\sqrt{4^2 + (p-3)^2} = 2\sqrt{13}$$

$$p^2 - 6p + 9 + 16 = 52$$

$$(p+3)(p-9) = 0$$

$$p = -3 \text{ OR } p = 9$$
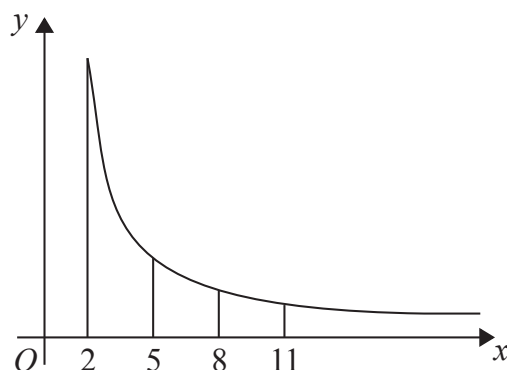
2.

Figure 1

Figure 1 shows a sketch of part of the graph of $y = \dfrac{12}{\sqrt{(x^2 - 2)}}$, $x \geqslant 2$

The table below gives values of y rounded to 3 decimal places.

x	2	5	8	11
y	8.485	2.502	1.524	1.100

(a) Use the trapezium rule with all the values of y from the table to find an approximate value, to 2 decimal places, for

$$\int_2^{11} \frac{12}{\sqrt{(x^2 - 2)}}\, dx$$

(4)

(b) Use your answer to part (a) to estimate a value for

$$\int_2^{11} \left(1 + \frac{6}{\sqrt{(x^2 - 2)}} \right) dx$$

(3)

ⓐ

$$A = \tfrac{1}{2} \times 3 \times (8.485 + 1.100 + 2(2.502 + 1.524))$$

$$= 26.4555$$

$$\simeq 26.46 \ (2dp)$$

ⓑ $\displaystyle\int_2^{11} 1 + \frac{6}{\sqrt{x^2-2}}\, dx = \int_2^{11} 1\, dx + \frac{1}{2}\int \frac{12}{\sqrt{x^2-2}}\, dx$

Question 2 continued

$$= \left[x \right]_2^{11} + \frac{1}{2} (26.4555)$$

$$= 11 - 2 + \frac{1}{2} (26.4555)$$

$$= 22.22775$$

$$\approx 22.23 \ (2dp)$$

(Total for Question 2 is 7 marks)

3. Given that

$$2\log_4(2x + 3) = 1 + \log_4 x + \log_4(2x - 1), \quad x > \frac{1}{2}$$

(a) show that

$$4x^2 - 16x - 9 = 0$$

(5)

(b) Hence solve the equation

$$2\log_4(2x + 3) = 1 + \log_4 x + \log_4(2x - 1), \quad x > \frac{1}{2}$$

(2)

(a)

$$\log_4(2x+3)^2 = \log_4 4 + \log_4 x + \log_4(2x-1)$$

$$(2x+3)^2 = 4(x)(2x-1)$$

$$4x^2 + 12x + 9 = 8x^2 - 4x$$

$$4x^2 - 16x - 9 = 0$$

as required.

(b)

$$(2x-9)(2x+1) = 0$$

$$x = \frac{9}{2} \quad \text{OR} \quad x = \frac{-1}{2}$$

(rejected)

$$\therefore \quad x = \frac{9}{2} \quad \text{only}$$

4. The curve with equation $y = 6^{1-x}$ meets the curve with equation $y = 3 \times 4^x$ at the point P.

Show that the x coordinate of P is $\dfrac{\log_{10} 2}{\log_{10} 24}$

(5)

$$6^{1-x} = 3(4^x)$$

$$\log_{10} 6^{1-x} = \log(3 \cdot 4^x)$$

$$(1-x)\log_{10} 6 = \log_{10} 3 + \log_{10} 4^x$$

$$\log_{10} 6 - x\log_{10} 6 = \log_{10} 3 + x\log_{10} 4$$

$$\log_{10} 6 - \log_{10} 3 = x(\log_{10} 4 + \log_{10} 6)$$

$$\log_{10} 2 = x\log_{10} 24$$

$$x = \frac{\log_{10} 2}{\log_{10} 24}$$

5. The temperature, $\theta\,°C$, inside an oven, t minutes after the oven is switched on, is given by

$$\theta = A - 180e^{-kt}$$

where A and k are positive constants.

Given that the temperature inside the oven is initially $18\,°C$,

(a) find the value of A.

(2)

The temperature inside the oven, 5 minutes after the oven is switched on, is $90\,°C$.

(b) Show that $k = p \ln q$ where p and q are rational numbers to be found.

(4)

Hence find

(a)

$$\text{when } t=0 \quad A=18$$

$$18 = A - 180(1)$$

$$18 = A - 180$$

$$A = 198$$

(b)

$$t=5, \quad A=90$$

$$90 = 198 - 180e^{-k(5)}$$

$$108 = 180e^{-5k}$$

$$e^{-5k} = \frac{3}{5}$$

$$-5k = \ln\left(\frac{3}{5}\right)$$

$$k = \frac{-1}{5}\ln\left(\frac{3}{5}\right)$$

Question 5 continued

$$K = \frac{1}{5} \ln\left(\frac{5}{3}\right)$$

$$P = \frac{1}{5}, \quad q = \frac{5}{3} \, \text{//}$$

(Total for Question 5 is 6 marks)

6.

(a) Express $\dfrac{9(4+x)}{16-9x^2}$ in partial fractions.

(3)

Given that

$$f(x) = \frac{9(4+x)}{16-9x^2}, \qquad x \in \mathbb{R}, \qquad -\frac{4}{3} < x < \frac{4}{3}$$

(b) express $\int f(x)\,dx$ in the form $\ln(g(x))$, where $g(x)$ is a rational function.

(4)

(a)

$$\frac{9(4+x)}{(4-3x)(4+3x)} \equiv \frac{A}{4-3x} + \frac{B}{4+3x}$$

$$36+9x \equiv A(4+3x) + B(4-3x)$$

$$4A + 4B = 36 \quad \text{—①}$$

$$3A - 3B = 9 \quad \text{—②}$$

$$A = 6, \quad B = 3$$

$$\Rightarrow \frac{9(4+x)}{16-9x^2} \equiv \frac{6}{4-3x} + \frac{3}{4+3x}$$

(b)

$$\int \frac{6}{4-3x}\,dx + \int \frac{-3}{4+3x}\,dx$$

$$= -2\int \frac{-3}{4-3x} + \int \frac{3}{4+3x}\,dx$$

$$= -2\ln|4-3x| + \ln|4+3x| + C$$

Question 6 continued

$$= \ln\left|(4-3x)^{\frac{1}{2}}\right| + \ln|4+3x| + C$$

$$= \ln\left|\frac{4+3x}{(4-3x)^2}\right| + C$$

$$\therefore |x| < \frac{4}{3}$$

$$= \ln\frac{4+3x}{(4-3x)^2} + \ln K$$

$$= \ln\left(\frac{K(4+3x)}{(4-3x)^2}\right) =$$

7.

$$f(x) = x\cos\left(\frac{x}{3}\right) \qquad x > 0$$

(a) Find $f'(x)$

(2)

(b) Show that the equation $f'(x) = 0$ can be written as

$$x = k\arctan\left(\frac{k}{x}\right)$$

where k is an integer to be found.

(2)

(c) Starting with $x_1 = 2.5$ use the iteration formula

$$x_{n+1} = k\arctan\left(\frac{k}{x_n}\right)$$

with the value of k found in part (b), to calculate the values of x_2 and x_6 giving your answers to 3 decimal places.

(2)

(d) Using a suitable interval and a suitable function that should be stated, show that a root of $f'(x) = 0$ is 2.581 correct to 3 decimal places.

(2)

(a)

$$f(x) = x\cos\left(\frac{x}{3}\right)$$

$$f'(x) = x\left(-\sin\frac{x}{3}\right)\left(\frac{1}{3}\right) + \cos\frac{x}{3}$$

(b)

$$f'(x) = 0$$

$$-\frac{x}{3}\sin\frac{x}{3} + \cos\frac{x}{3} = 0$$

$$\frac{x}{3}\sin\frac{x}{3} = \cos\frac{x}{3}$$

$$\frac{3}{x} = \tan \frac{x}{3}$$

$$\arctan\left(\frac{3}{x}\right) = \frac{x}{3}$$

$$3 \arctan \frac{3}{x} = x$$

(c)

$$x_{n+1} = 3 \arctan\left(\frac{3}{x}\right)$$

$$x_1 = 2.5$$

$$x_2 = 2.628174152$$

$$x_2 \simeq 2.628 \quad (3 dp)$$

$$x_3 = 2.554102174$$

$$x_4 = 2.596526871$$

$$x_5 = 2.572101407$$

$$x_6 = 2.586122337$$

$X_6 = 2.586 (3dp)$

(d)

$$f'(x) = -\frac{X}{3} \sin \frac{X}{3} + \cos \frac{X}{3}$$

$$f'(2.5815) = -0.00034568972 \quad < 0$$

$$f'(2.5805) = 0.0003467157 \quad > 0$$

change in sign, continuous

$$f'(x) = 0 \quad \text{has root } \alpha$$

$$2.5805 < \alpha < 2.5815$$

8. (i) Find

$$\int \frac{(3x+2)^2}{4\sqrt{x}} \, dx \qquad\qquad x > 0$$

giving your answer in simplest form.

(5)

(ii) A curve C has equation $y = f(x)$.

Given

- $f'(x) = x^2 + ax + b$ where a and b are constants
- the y intercept of C is -8
- the point $P(3,-2)$ lies on C
- the gradient of C at P is 2

find, in simplest form, $f(x)$.

(6)

ⓐ

$$\int \frac{(3x+2)^2}{4\sqrt{x}} \, dx = \int \frac{9x^2 + 12x + 4}{4\sqrt{x}} \, dx$$

$$= \int \frac{9}{4}x^{\frac{3}{2}} + 3x^{\frac{1}{2}} + x^{-\frac{1}{2}} \, dx$$

$$= \frac{9}{4}x^{\frac{5}{2}}\left(\frac{2}{5}\right) + 3x^{\frac{3}{2}}\left(\frac{2}{3}\right) + x^{\frac{1}{2}}(2) + c$$

$$= \frac{9}{10}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c \;\;//$$

ⓑ

$$\int f'(x) \, dx = f(x)$$

$$\int x^2 + ax + b \, dx = \frac{x^3}{3} + \frac{ax^2}{2} + bx + c = f(x)$$

$$x=0 \quad f(x) = -8 \implies c = -8$$

$x=3, \ y=-2, \ f'(x)=2$

$x^2 + ax + b = f'(x)$

$9 + 3a + b = 2 \qquad -①$

$\frac{x^3}{3} + \frac{ax^2}{2} + bx + C = f(x)$

$\frac{27}{3} + \frac{a}{2}a + 3b + C = -2 \qquad -②$

$3a + b = -7$

$\frac{a}{2}a + 3b = -2 - \frac{27}{3} - (-8)$

$\Rightarrow a = -4 \quad b = 5 \quad //$

$\Rightarrow f(x) = \frac{x^3}{3} - 2x^2 + 5x - 8 \quad //$

9. (a) Prove

$$\frac{\cos 3\theta}{2\sin\theta} + \frac{\sin 3\theta}{2\cos\theta} \equiv \cot 2\theta \qquad \theta \neq \frac{n\pi}{2} \quad n \in \mathbb{Z}$$

(4)

(b) Hence solve, for $0 < x < \frac{\pi}{2}$

$$\frac{\cos 3x}{2\sin x} + \frac{\sin 3x}{2\cos x} = 5\cos 2x$$

giving your answers to 3 decimal places where appropriate.

(4)

(a)

$$LHS = \frac{\cos 3\theta (\cos\theta) + \sin 3\theta (\sin\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{\cos(3\theta - \theta)}{\sin 2\theta}$$

$$= \frac{\cos 2\theta}{\sin 2\theta}$$

$$= \cot 2\theta$$

$$= RHS$$

(b) $\cot 2x = 5\cos 2x \qquad 0 < x < \frac{\pi}{2}$

24

$$\frac{\cos 2x}{\sin 2x} = 5\cos 2x$$

$$\cos 2x \left(5 - \frac{1}{\sin 2x}\right) = 0$$

$$\cos 2x = 0 \quad \text{OR} \quad \frac{1}{\sin 2x} = 5$$

$$2x = \frac{\pi}{2} \quad \text{OR} \quad \sin 2x = \frac{1}{5}$$

$$x = \frac{\pi}{4} \quad \text{OR}$$

$$2x = 0.20135 \qquad 2x = \pi - 0.20135$$

$$x = 0.101 \qquad x = 1.470$$
$$(3dp) \qquad (3dp)$$

$$x = \frac{\pi}{4}, \; 0.101, \; 1.470$$

(Total for Question 9 is 8 marks)

10.
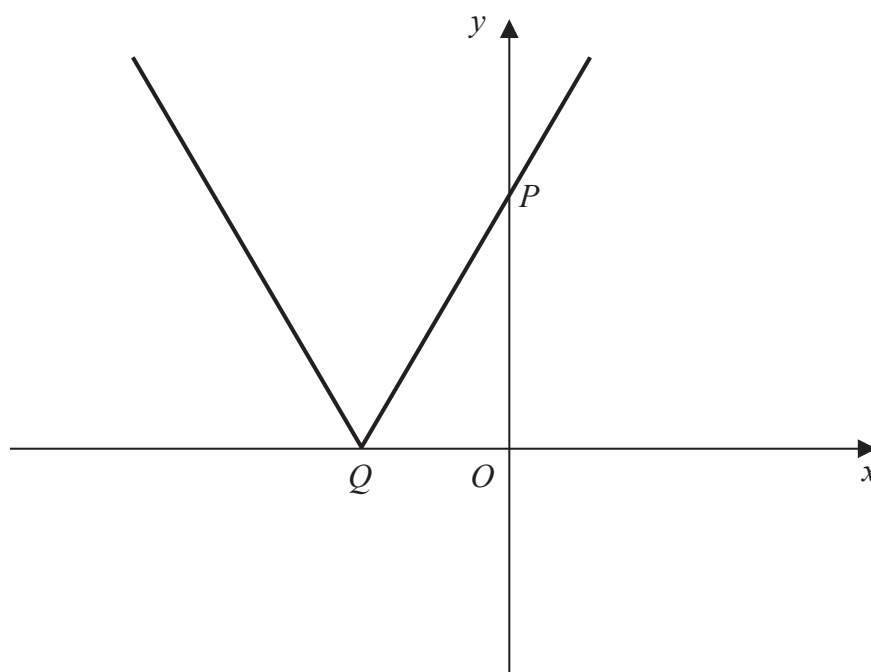
Figure 1

Figure 1 shows a sketch of the graph with equation $y = |4x + 10a|$, where a is a positive constant.

The graph cuts the y-axis at the point P and meets the x-axis at the point Q as shown.

(a) (i) State the coordinates of P. $(0, 10a)$

(ii) State the coordinates of Q. $\left(-\frac{5}{2}a, 0\right)$

(2)

(b) A copy of Figure 1 is shown on page 27. On this copy, sketch the graph with equation

$$y = |x| - a$$

Show on the sketch the coordinates of each point where your graph cuts or meets the coordinate axes.

(2)

(c) Hence, or otherwise, solve the equation

$$|4x + 10a| = |x| - a$$

giving your answers in terms of a.

(3)

$y = |4x + 10a|$

gradient $= -4/4$

$y = |x| - a$

gradient $= -1/1$

$\left(-\frac{5}{2}a, 0\right)$ $-a$ a

$-c$
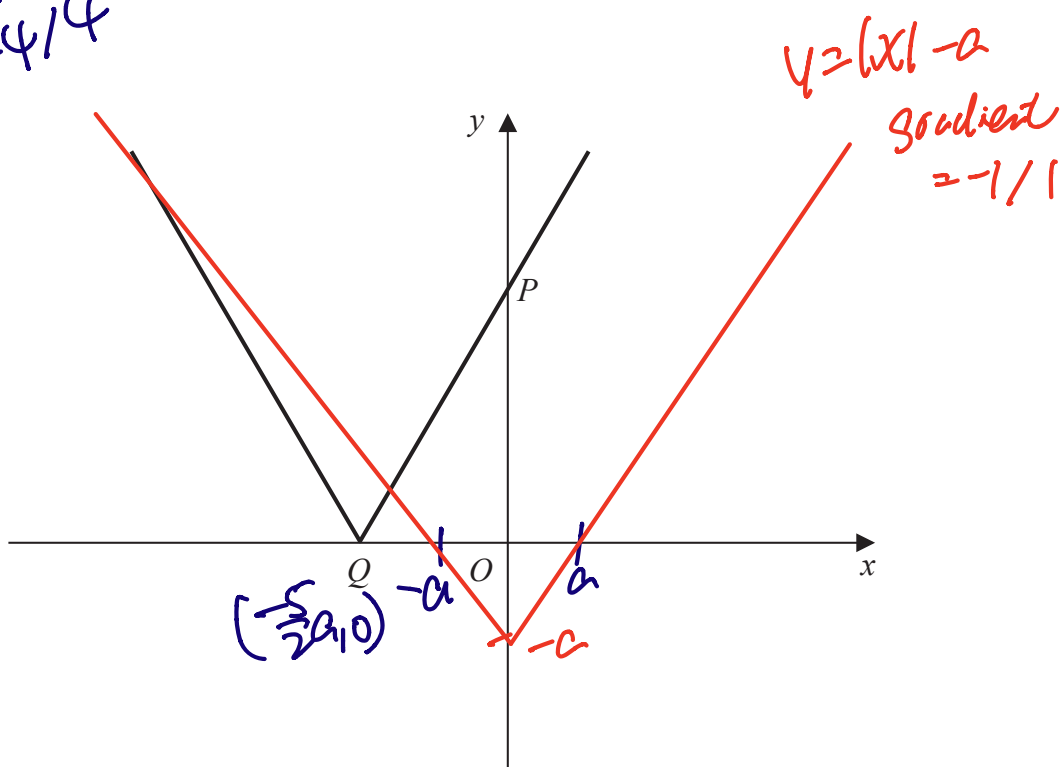
Figure 1

$x = 0, \ y = 10a$ $\Rightarrow \ P : (0, 10a)$

$y = 0$ $4x + 10a = 0$ $a > 0$

$x = \dfrac{-10a}{4}$

$x = -\dfrac{5}{2}a$ $\Rightarrow \ Q : \left(\dfrac{-5}{2}a, 0\right)$

$$|4x + 10a| = |x| - a$$

intersect at $x < 0$

when

$$4x + 10a = -x - a \quad \text{OR} \quad -4x - 10a = -x - a$$

$$5x = -11a \qquad\qquad -3x = 9a$$

$$x = \frac{-11}{5}a \qquad \text{OR} \qquad x = -3a$$
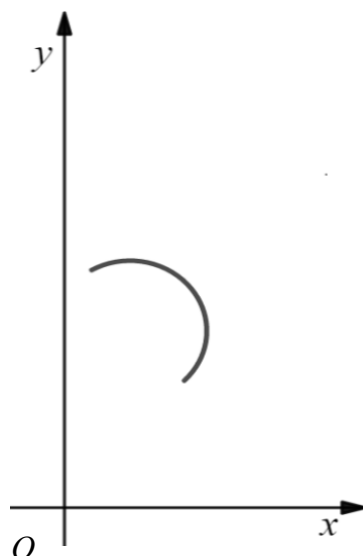
11.

Figure 1

The curve C has parametric equations

$$x = 3 + 2\sqrt{3}\cos t, \quad y = 5\sqrt{3} + 2\sqrt{3}\sin t, \qquad -\frac{\pi}{4} \leq t \leq \frac{2\pi}{3}$$

A sketch of C is shown in Figure 1.

(a) Show that all points on C satisfy $(x - 3)^2 + (y - 5\sqrt{3})^2 = 12$.

(2)

For curve C,

(b) (i) state the range of x,

    (ii) state the range of y.

(2)

The point P lies on C.

Given the line with equation $y = mx + 12\sqrt{3}$, where m is a constant, intersects C at P,

(c) state the range of m, writing your answer using set notation.

(6)

The points $(0, 0)$, $(0, 12\sqrt{3})$ and P form a triangle.

(d) (i) Find the largest possible area of the triangle

    (ii) Find the smallest possible area of the triangle.

(2)

$$X = 3 + 2\sqrt{3}\cos t \qquad y = 5\sqrt{3} + 2\sqrt{3}\sin t$$

LHS
$$= (x-3)^2 + (y - 5\sqrt{3})^2$$

$$= (3 + 2\sqrt{3}\cos t - 3)^2 + (5\sqrt{3} + 2\sqrt{3}\sin t - 5\sqrt{3})^2$$

$$= (2\sqrt{3}\cos t)^2 + (2\sqrt{3}\sin t)^2$$

$$= 12(\cos^2 t + \sin^2 t)$$

$$= 12$$

$$= \text{RHS}$$

$x = 3 + 2\sqrt{3}\cos t, \quad y = 5\sqrt{3} + 2\sqrt{3}\sin t, \qquad -\dfrac{\pi}{4} \le t \le \dfrac{2\pi}{3}$
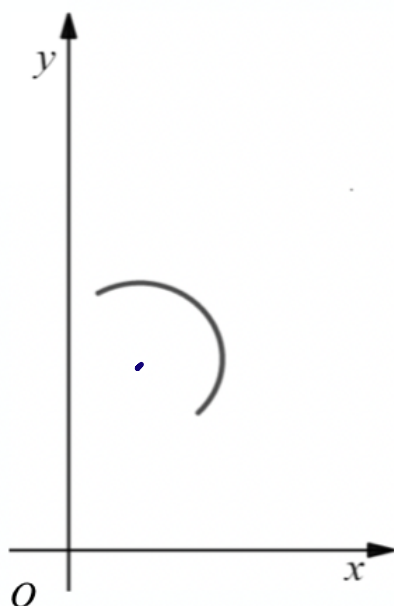
$(x-3)^2 + (y - 5\sqrt{3})^2 = 12.$

(b) (i) state the range of x,

(ii) state the range of y.

x:

max at $3 + 2\sqrt{3}\,(1)$

min at $3 - 2\sqrt{3}\,(-1/2)$

[EXE]:Show coordinates
Y1=5(√3)+2(√3)cos x,[((-π)⌐4),((2π)⌐3
dY/dX=0
X=0, ... ,Y=12.12435565, ,x
MAX

13
12
11
10
9
8
7
6
5
4
3

$$3 - \sqrt{3} \le X \le 3 + 2\sqrt{3}$$

```
|📋| Math Rad Norm2  d/c a+bi
3+2√3 cos  (−π÷4)
                        3+√6
3+2√3 cos  (2π÷3)
                        3−√3
3+2√3
                        3+2√3
| TOP || BOTTOM ||PageUp||PageDown|
```

```
|📋| Math Rad Norm2  d/c a+bi
3+2√3 cos  (−π÷4)
            5.449489743
3+2√3 cos  (2π÷3)
            1.267949192
3+2√3
            6.464101615
| TOP || BOTTOM ||PageUp||PageDown|
```

$$y = 5\sqrt{3} + 2\sqrt{3}\sin t$$

$$\text{max at} \quad y = 5\sqrt{3} + 2\sqrt{3}\,(1)$$

$$\text{min at} \quad y = 5\sqrt{3} + 2\sqrt{3}\left(\frac{-\sqrt{2}}{2}\right)$$

```
|📋| Math Rad Norm2  d/c a+bi
5√3 +2√3 sin  (−π÷4)
            6.210764295
5√3 +2√3 sin  (2π÷3)
            11.66025404
5√3 +2√3
            12.12435565
  (2π÷3)
| TOP || BOTTOM ||PageUp||PageDown|
```

```
|📋| Math Rad Norm2  d/c a+bi
5√3 +2√3 sin  (−π÷4)
            −√6 +5√3
5√3 +2√3 sin  (2π÷3)
            3+5√3
5√3 +2√3
            7√3
| TOP || BOTTOM ||PageUp||PageDown|
```

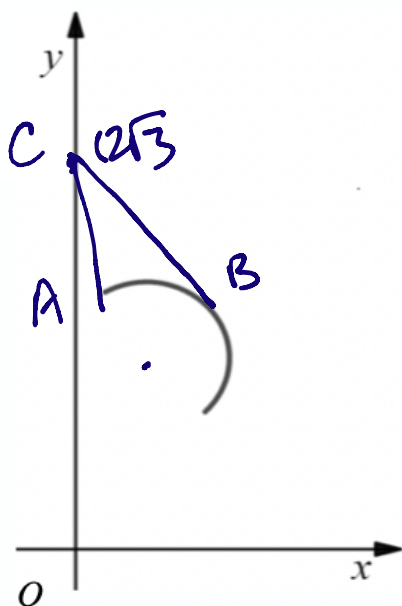$$5\sqrt{3} - \sqrt{6} \leq y \leq 7\sqrt{3}$$

The point P lies on C.

$$C = (0, 12\sqrt{3})$$

Given the line with equation $y = mx + 12\sqrt{3}$, where m is a constant, intersects C at P,

(c)   state the range of m, writing your answer using set notation.

(6)

A:

when $x = 3 - \sqrt{3}$, $y = 3 + 5\sqrt{3}$

gradient AC

$$\frac{3 + 5\sqrt{3} - 12\sqrt{3}}{3 - \sqrt{3}} = m$$

$$m = -2 - 3\sqrt{3}$$

gradient BC: Sub line $y = mx + 12\sqrt{3}$ into

$$(x-3)^2 + (y - 5\sqrt{3})^2 = 12$$

$$x^2 - 6x + 9 + (mx + 12\sqrt{3} - 5\sqrt{3})^2 = 12$$

$$x^2 - 6x + 9 + m^2x^2 + 14\sqrt{3}mx + 49(3) = 12$$

(Total for Question 11 is 12 marks)

$$(m^2+1)x^2 + (14\sqrt{3}m-6)x + 144 = 0$$

$$(14\sqrt{3}m-6)^2 - 4(m^2+1)(144) = 0$$

$$588m^2 - 168\sqrt{3}m + 36 - 576m^2 - 576 = 0$$

$$12m^2 - 168\sqrt{3}m - 540 = 0$$

$$\overset{\alpha}{\downarrow} \qquad \overset{\beta}{\downarrow}$$

$$m = 15\sqrt{3} \qquad OR \qquad m = -\sqrt{3}$$
$$(HCi)$$

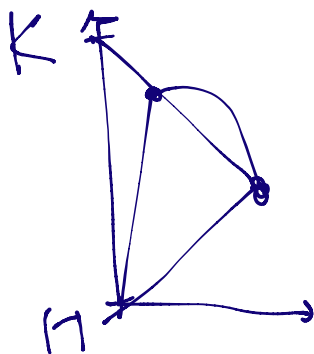$$\left\{ x: \quad -2-3\sqrt{3} \le x \le -\sqrt{3} \right\}$$

The points (0, 0), (0, 12√3) and *P* form a triangle.
H   K

(d) (i) Find the largest possible area of the triangle

(ii) Find the smallest possible area of the triangle.

(2)

Base is fixed,
therefore =
    largest possible
$$= \tfrac{1}{2}(12\sqrt{3})(3+2\sqrt{3}) = 36+18\sqrt{3}$$

smallest possible

$$= \frac{1}{2}(12\sqrt{3})(3-\sqrt{3}) = -18+18\sqrt{3}$$

12. The circle C has equation

$$x^2 + y^2 + 6x - 4y - 14 = 0$$

(a) Find

    (i) the coordinates of the centre of C,

    (ii) the exact radius of C.

(3)

The line with equation $y = k$, where k is a constant, is a tangent to C.

(b) Find the possible values of k.

(2)

The line with equation $y = p$, where p is a negative constant, is a chord of C.

Given that the length of this chord is 4 units,
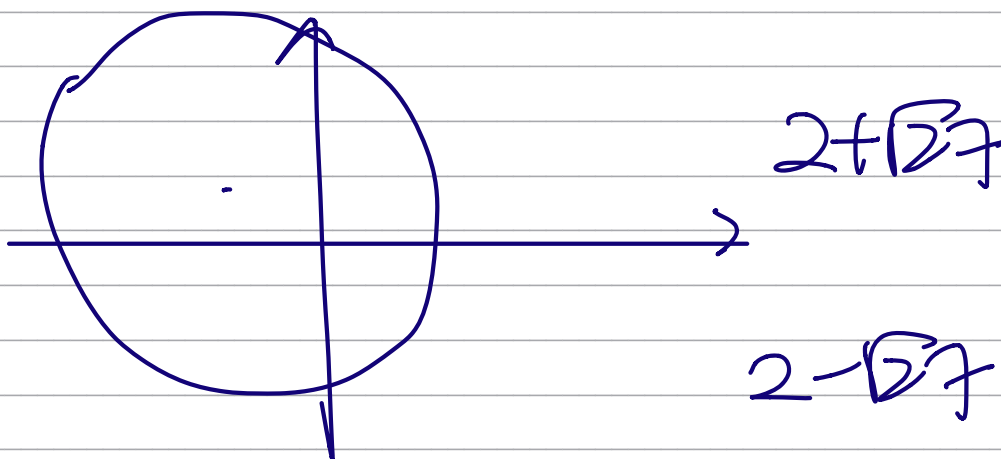
(c) find the value of p.

(3)

$$(x+3)^2 + (y-2)^2 - 9 - 4 = 14$$
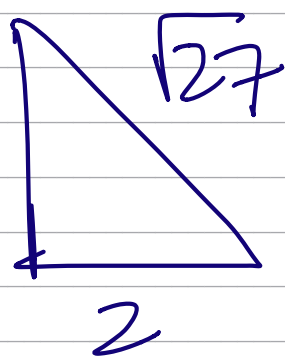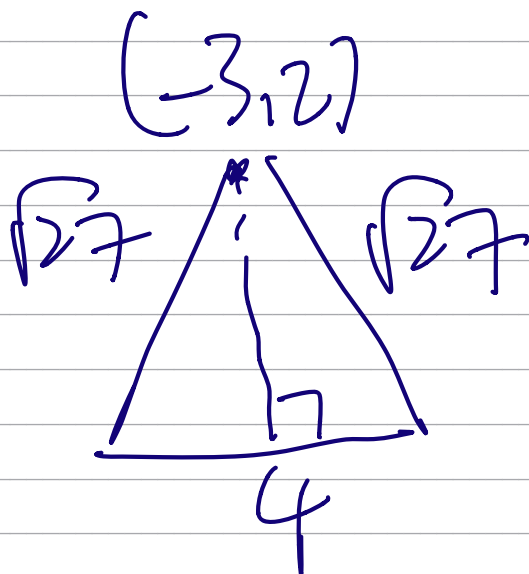
$$(x+3)^2 + (y-2)^2 = 27$$

centre $(-3, 2)$

radius $(\sqrt{27})$

(b)

$2 + \sqrt{27}$

$2 - \sqrt{27}$

$$K = 2 \pm \sqrt{27}$$

(c)

$$(-3, 2)$$

$$a^2 + b^2 = c^2$$

$$a^2 + 4 = 27$$

$$a^2 = 23$$

$$a = \sqrt{23}$$

$$P = 2 - \sqrt{23}$$

13   (a) Prove that the sum of the first n terms of an arithmetic series is given by the formula

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

where a is the first term of the series and d is the common difference between the terms.

(4)

(b) Find the sum of the integers which are divisible by 7 and lie between 1 and 500

(3)

(a)

$$S_n = a + a+d + a+2d + \cdots + a+(n-1)d$$

$$S_n = a+(n-1)d + a+(n-2)d + \cdots + a+d + a$$

$$2S_n = [2a+(n-1)d]n$$

$$S_n = \frac{n}{2}(2a+(n-1)d)$$

(b)

$$7 + 14 + \cdots + 497 = S$$

$$7(1) + 7(2) + \cdots + 7(71) = S$$

$$\frac{[7 + 7(71)](71)}{2} = 17892 \text{ //}$$

14. Prove by contradiction that, if a, b are positive real numbers, then $a + b \geqslant 2\sqrt{ab}$

(4)

a, b are positive real numbers,

Assume $a + b < 2\sqrt{ab}$

$(a+b)^2 < 4ab$

$a^2 + 2ab + b^2 < 4ab$

$a^2 - 2ab + b^2 < 0$

$(a-b)^2 < 0$

⚡

contradiction,

therefore $a + b$ NOT $\leqslant 2\sqrt{ab}$

therefore if $a, b \in \mathbb{R}^+$

$a + b \geqslant 2\sqrt{ab}$