7.

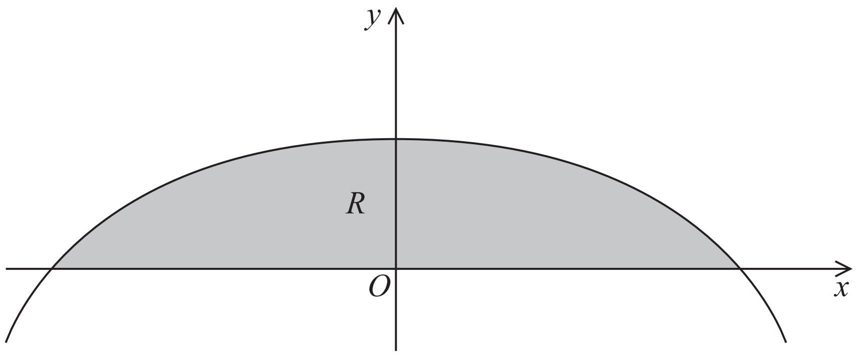

Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = \frac{\cos x}{1 + \cos x}$$

The region R, shown shaded in Figure 1, is bounded by the curve and the x-axis.

Use calculus and the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that the exact area of region R is $a\pi + b$, where a and b are constants to be determined.

(9)

Question	Scheme	Marks	AOs
7	$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1 + \tan^2\left(\frac{x}{2}\right)\right) = \frac{1}{2}(1 + t^2)$	B1	2.1
	$\int \frac{\frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t$	M1 A1	3.1a 1.1b
	$\int \frac{1 - t^2}{1 + t^2}\,\mathrm{d}t = \int \frac{-(1+t^2)+2}{1+t^2}\mathrm{d}t = \int -1 + \frac{2}{1+t^2}\mathrm{d}t$	M1	2.1
	$= -t + 2\arctan(t)$	M1 A1	1.1b 1.1b
	Limits seen $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ or $t = 1$ and $t = -1$ or $x = \frac{\pi}{2}$ and $x = 0$ or $t = 1$ and $t = 0$ and using 2 times integral	B1	3.1a
	Substitutes corresponding limits and subtracts the correct way around	M1	1.1b
	$\pi - 2$ or $a = 1$ and $b = -2$	A1	1.1b
		(9)	
			(9 marks)

Notes:

B1: $\frac{dt}{dx} = \frac{1}{2}(1 + t^2)$
M1: A complete strategy find to integrate the curve
A1: Correct un-simplified integral including dt
M1: Rearranging the integral into the form $a + \frac{b}{1+t^2}$ could be from the use of long division
M1: Integrates to the form $\beta t + \gamma\arctan(t)$
A1: $-t + 2\arctan(t)$
B1: A complete strategy to find the values of the constant a and b
$x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ or $t = 1$ and $t = -1$
If using 2 times integral $x = \frac{\pi}{2}$ and $x = 0$ or $t = 1$ and $t = 0$
M1: Correct use of the limits $t = 1$ and $t = -1$ or substitutes back $t = \tan\left(\frac{x}{2}\right)$ and correct use of the limits $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$
A1: Area = $\pi - 2$

8. [*In this question you may assume the t-formulae for* $\cos x$ *and* $\frac{dx}{dt}$ *without proof.*]

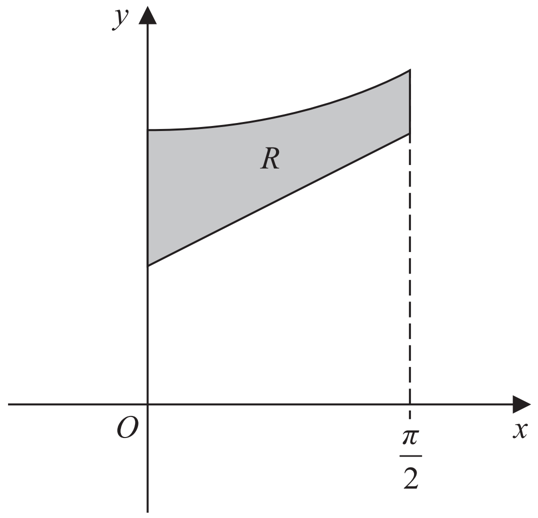

Figure 3

An engineer is designing a part for a motorcycle.

The part is modelled by the solid formed by rotating the region R, shown shaded in Figure 3, through 2π radians about the x-axis.

Region R is bounded by the y-axis, the line with equation $\pi y = 2x + \pi$, the line with equation $x = \frac{\pi}{2}$ and the curve with equation

$$y = \sqrt{\frac{12}{2 + \cos x}}$$

where x and y are measured in centimetres.

The part is to be made from steel of density $8\,\text{g cm}^{-3}$

(a) Using the substitution $t = \tan\left(\frac{x}{2}\right)$ and algebraic integration, find, according to the model, the exact mass in grams of the part, giving your answer in the form

$$\frac{4}{3}\pi^2\left(a\sqrt{3} + b\right)$$

where a and b are integers to be determined.

(11)

(b) Give a reason why the mass of steel predicted by the model may not be an accurate value for the mass of steel that would be used to make the part.

(1)

Q8	Scheme	Marks	AOs
(a)	$(\pi)\int y^2 \, dx = (\pi)\int \left(\sqrt{\frac{12}{2+\cos x}}\right)^2 dx = (\pi)\int \frac{12}{2+\cos x}\, dx$	M1	3.1a
	$t = \tan\frac{x}{2} \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$ or $\frac{dx}{dt} = \frac{2}{1+t^2}$	B1	1.2
	$\int \frac{12}{2+\cos x}\, dx = 12\int \frac{1}{2+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\, dt$ $= 12\int \frac{1}{\frac{2(1+t^2)+1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\, dt$ $= 12\int \frac{1+t^2}{t^2+3} \times \frac{2}{1+t^2}\, dt$	M1	2.1
	$= \int \frac{24}{t^2+3}\, dt$	A1	1.1b
	$\Rightarrow \frac{24}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}}$ oe e.g., $8\sqrt{3}\tan^{-1}\frac{\sqrt{3}t}{3}$	M1 A1	1.1b 1.1b
	$x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\frac{\pi}{2}}{2} = \tan\frac{\pi}{4} = 1,\ x = 0 \Rightarrow t = \tan 0 = 0$ V of revolution of curve $= \frac{24\pi}{\sqrt{3}}\left[\arctan\frac{t}{\sqrt{3}}\right]_0^1$ or $\frac{24\pi}{\sqrt{3}}\left[\arctan\frac{\tan\frac{x}{2}}{\sqrt{3}}\right]_0^{\frac{\pi}{2}}$ $= \frac{24\pi}{\sqrt{3}}\arctan\frac{1}{\sqrt{3}} - 0$ or $\frac{24\pi}{\sqrt{3}}\arctan\frac{\tan\frac{\pi}{4}}{\sqrt{3}} - 0$ $= \frac{24\pi}{\sqrt{3}}\left(\frac{\pi}{6}\right) = \frac{4}{\sqrt{3}}\pi^2 = \frac{4\sqrt{3}}{3}\pi^2$	dM1	1.1b

	By mensuration: $y = \frac{2}{\pi}x + 1 \Longrightarrow y = 0, x = -\frac{\pi}{2}$ and $x = 0, y = 1$ and $x = \frac{\pi}{2}, y = 2$ $\Longrightarrow$ large cone: $h = \pi, r = 2$; small cone: : $h = \frac{\pi}{2}, r = 1$ $V_{\text{frustum}} = \frac{1}{3}\pi\left(2^2\pi - 1^2\left(\frac{\pi}{2}\right)\right) = \frac{1}{3}\pi\left(4\pi - \frac{\pi}{2}\right) = \frac{7}{6}\pi^2$ Or by integration: $V_{\text{frustum}} = \int_0^{\frac{\pi}{2}} \left(\frac{2}{\pi}x + 1\right)^2 \mathrm{d}x$ $= \int_0^{\frac{\pi}{2}} \left(\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x + 1\right)\mathrm{d}x$ $= \pi\left[\frac{4}{3\pi^2}x^3 + \frac{2}{\pi}x^2 + x\right]_0^{\frac{\pi}{2}}$ $= \pi\left[\frac{4}{3\pi^2}\left(\frac{\pi}{2}\right)^3 + \frac{2}{\pi}\left(\frac{\pi}{2}\right)^2 + \frac{\pi}{2}\right]_0^{\frac{\pi}{2}} = \pi\left(\frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{7}{6}\pi^2$	**M1** **A1**	3.1b 2.2a
	Volume of part = $\left(\frac{4\sqrt{3}}{3} - \frac{7}{6}\right)\pi^2 = \frac{8\sqrt{3} - 7}{6}\pi^2$ mass = density x volume = 8 x volume = $\frac{32\sqrt{3} - 28}{3}\pi^2$	**M1**	3.4
	$= \frac{4}{3}\pi^2(8\sqrt{3} - 7)$ (g)	**A1**	2.1
		(11)	
(b)	e.g., The measurements of the part may not be accurate/ when the part is made there may be deviations from the shape in the model The steel may not be of uniform density	**B1**	3.5b
		(1)	
			(12 marks)
			TOTAL FOR PAPER IS 75 MARKS

Notes

(a)

M1: Any use of the volume of revolution formula condoning a missing or incorrect π (could be for the frustum).

B1: Either correct cos x or $\frac{dx}{dt}$ in terms of t seen or used

M1: Substitutes their cos x and $\frac{dx}{dt}$ and simplifies expression to a single fraction in t

A1: Correct integral

M1: Integrates to obtain arctan term only

A1: Correct integration

dM1: Correctly uses correct limits – dependent on previous method mark

M1: A complete method to find $V_{frustum}$ – allow slips but correct formulae must be used/integration should produce expression of the correct form

A1: $\frac{7}{6}\pi^2$ only

M1: Uses model density to calculate 8(volume from curve – volume of frustum)

A1: $\frac{4}{3}\pi^2\,(8\sqrt{3}-7)$ only

(b)

B1: Any acceptable reason in context

8.

$$f(x) = \frac{3}{13 + 6\sin x - 5\cos x}$$

Using the substitution $t = \tan\left(\frac{x}{2}\right)$

(a) show that $f(x)$ can be written in the form

$$\frac{3(1+t^2)}{2(3t+1)^2 + 6}$$

(3)

(b) Hence solve, for $0 < x < 2\pi$, the equation

$$f(x) = \frac{3}{7}$$

giving your answers to 2 decimal places where appropriate.

(5)

(c) Use the result of part (a) to show that

$$\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} f(x)\,dx = K\left(\arctan\left(\frac{\sqrt{3}-9}{3}\right) - \arctan\left(\frac{\sqrt{3}+3}{3}\right) + \pi\right)$$

where K is a constant to be determined.

(8)

Question	Scheme	Marks	AOs
8(a)	$\mathrm{f}(x) = \dfrac{3}{13+6\times\dfrac{2t}{1+t^2}-5\times\dfrac{1-t^2}{1+t^2}}$	**M1**	1.1b
	$= \dfrac{3(1+t^2)}{13(1+t^2)+12t-5(1-t^2)}$	**M1**	1.1b
	$= \dfrac{3(1+t^2)}{18t^2+12t+8} \Rightarrow$ for example $\dfrac{3(1+t^2)}{2(9t^2+6t+1)+6}$ or $\dfrac{3(1+t^2)}{2\left[(3t+1)^2-1\right]+8}$ $\Rightarrow \dfrac{3(1+t^2)}{2(3t+1)^2+6}$ *	**A1***	2.1
		(3)	
(b)	$\mathrm{f}(x) = \dfrac{3}{7} \Rightarrow \dfrac{3(1+t^2)}{2(3t+1)^2+6} = \dfrac{3}{7} \Rightarrow 21+21t^2 = 54t^2+36t+24$ $\Rightarrow 11t^2+12t+1=0$	**M1**	1.1b
	$\Rightarrow (11t+1)(t+1)=0 \Rightarrow t = ...$	**M1**	1.1b
	$t=-1,\ t=-\dfrac{1}{11}$	**A1**	1.1b
	$\Rightarrow x = 2\arctan(\text{"}\mathit{their}\ t\text{"})+2\pi$ for a negative t	**dM1**	3.1a
	$x = \dfrac{3\pi}{2}$ or awrt 4.71 and awrt $x = 6.10$	**A1**	1.1b
		(5)	
(c)	$\int \mathrm{f}(x) = \int \dfrac{3(1+t^2)}{2(3t+1)^2+6}\times\dfrac{2}{1+t^2}\,\mathrm{d}t = \int \dfrac{3}{(3t+1)^2+3}\,\mathrm{d}t$	**B1**	2.1
	$= K\arctan\left(M(3t+1)\right)$	$u=(3t+1) \Rightarrow K\arctan(Mu)$	**M1**	1.1b
	$= \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{3t+1}{\sqrt{3}}\right)$	$= \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{u}{\sqrt{3}}\right)$	**A1**	1.1b
	$\int_{\frac{\pi}{3}}^{\frac{4\pi}{3}} \mathrm{f}(x)\,\mathrm{d}x = \int_{\frac{\pi}{3}}^{\pi} \mathrm{f}(x)\,\mathrm{d}x + \int_{\pi}^{\frac{4\pi}{3}} \mathrm{f}(x)\,\mathrm{d}x$ $= \int_{\frac{\sqrt{3}}{3}}^{\infty} ...\,\mathrm{d}t + \int_{-\infty}^{-\sqrt{3}} ...\,\mathrm{d}t$ or $\int_{\sqrt{3}+1}^{\infty} ...\,\mathrm{d}u + \int_{-\infty}^{1-3\sqrt{3}} ...\,\mathrm{d}u$	**B1**	3.1a
	$= \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{3(-\sqrt{3})+1}{\sqrt{3}}\right) - \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{3\left(\dfrac{\sqrt{3}}{3}\right)+1}{\sqrt{3}}\right) + ...$	**M1**	1.1b

	$=\frac{\sqrt{3}}{3}\left(\arctan\left(\frac{\sqrt{3}-9}{3}\right)-\arctan\left(\frac{\sqrt{3}+3}{3}\right)\right)+\ldots$	**A1**	1.1b
	$=\ldots+\lim_{t\to\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{3t+1}{\sqrt{3}}\right)-\lim_{t\to-\infty}\frac{1}{\sqrt{3}}\arctan\left(\frac{3t+1}{\sqrt{3}}\right)=\ldots+\frac{\pi}{2\sqrt{3}}-\left(-\frac{\pi}{2\sqrt{3}}\right)$	**M1**	3.1a
	$=\frac{\sqrt{3}}{3}\left(\arctan\left(\frac{\sqrt{3}-9}{3}\right)-\arctan\left(\frac{\sqrt{3}+3}{3}\right)+\pi\right)$	**A1**	2.1
		(8)	
			(16 marks)

Notes:

(a)

M1: Uses one correct substitution

M1: Both substitutions correct and attempts to multiply through numerator and denominator by $1+t^2$.

A1*: Completes to the correct expression with no errors seen. Must see an intermediate step simplifying the denominator – most likely one of the ones seen in the scheme.

(b)

M1: Equates the result in (a) to $\frac{3}{7}$ and simplifies to a 3TQ

M1: Solves their equation by any valid means.

A1: Correct values for t

dM1: Dependent on first method mark. Applies the correct process to find at least one value for x from a negative value for t. (If two positive values are found in error, this mark cannot be scored.)

A1: Both answers correct and no others in range.

(c)

B1: Applies the substitution including the use of $\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$

M1: Attempts the integration to achieve $K\arctan\left(M(1+3t)\right)$ or $K\arctan\left(Mu\right)$ if using a substitution of $u=\left(3t+1\right)$.

May use substitution $3t+1=\sqrt{3}\tan\theta \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\theta}=\frac{\sqrt{3}}{3}\sec^2\theta \Rightarrow \int\frac{\sqrt{3}}{3}\mathrm{d}\theta=\frac{\sqrt{3}}{3}\theta=\frac{\sqrt{3}}{3}\arctan\left(\frac{3t+1}{\sqrt{3}}\right)$

A1: Correct integral.

B1: Changes the limits and splits the integral around π

M1: Applies their limits '$1/\sqrt{3}$' and '$-\sqrt{3}$' to their integrand.

A1: Correct "arctan" expressions.

M1: Correct work to evaluate the $\pm\infty$ limits

A1: Fully correct solution.

5.

$$I = \int \frac{1}{4\cos x - 3\sin x}\,\mathrm{d}x \qquad 0 < x < \frac{\pi}{4}$$

Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to show that

$$I = \frac{1}{5}\ln\left(\frac{2 + \tan\left(\frac{x}{2}\right)}{1 - 2\tan\left(\frac{x}{2}\right)}\right) + k$$

where k is an arbitrary constant.

(8)

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Question	Scheme	Marks	AOs
5	$4\cos x - 3\sin x = 4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)$	B1	1.1a
	$\frac{dt}{dx} = \frac{1+t^2}{2}$ or $\frac{dx}{dt} = \frac{2}{1+t^2}$ or $dx = \frac{2dt}{1+t^2}$ or $dt = \frac{1+t^2}{2}dx$ oe	B1 **M1 on ePEN**	2.1
	$\int \frac{1}{4\cos x - 3\sin x}dx = \int \frac{1}{4\left(\frac{1-t^2}{1+t^2}\right) - 3\left(\frac{2t}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$	M1	2.1
	$= \int \frac{2}{4-4t^2-6t}(dt)$ or $\int \frac{1}{2-2t^2-3t}(dt)$ or $\int \frac{-1}{2t^2+3t-2}(dt)$ etc.	A1	1.1b
	$\frac{-2}{4t^2+6t-4} = \frac{-1}{(t+2)(2t-1)} = \frac{A}{(t+2)} + \frac{B}{(2t-1)}$ $\frac{-1}{(t+2)(2t-1)} = \frac{1}{5(t+2)} + \frac{2}{5(1-2t)}$	M1	3.1a
	$\Rightarrow I = \frac{1}{5}\int \frac{1}{(t+2)} - \frac{2}{(2t-1)}(dt)$ or equivalent	A1	1.1b
	$= \frac{1}{5}\int \frac{1}{(t+2)} - \frac{2}{(2t-1)}dt = \frac{1}{5}\ln(t+2) - \frac{1}{5}\ln(1-2t)(+k)$	A1	1.1b
	$= \frac{1}{5}\ln\left(\frac{2+t}{1-2t}\right)(+k) = \frac{1}{5}\ln\left(\frac{2+\tan\left(\frac{x}{2}\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + k$ *	A1*	2.1
		(8)	
	Alternative for final 4 marks:		
	$= \int \frac{2}{4-4t^2-6t}(dt) = -\frac{1}{2}\int \frac{1}{t^2+\frac{3}{2}t-1}(dt) = -\frac{1}{2}\int \frac{1}{\left(t+\frac{3}{4}\right)^2 - \frac{25}{16}}(dt)$ or e.g. $\int \frac{1}{\frac{25}{8} - 2\left(t+\frac{3}{4}\right)^2}(dt)$	M1 A1	3.1a 1.1b
	$-\frac{1}{2} \times \frac{1}{2} \times \frac{4}{5}\ln\left(\frac{t+\frac{3}{4}-\frac{5}{4}}{t+\frac{3}{4}+\frac{5}{4}}\right)(+c)$	A1	1.1b
	$-\frac{1}{5}\ln\left\lvert\frac{\tan\left(\frac{x}{2}\right)-\frac{1}{2}}{\tan\left(\frac{x}{2}\right)+2}\right\rvert + c = \frac{1}{5}\ln\left\lvert\frac{\tan\left(\frac{x}{2}\right)+2}{\tan\left(\frac{x}{2}\right)-\frac{1}{2}}\right\rvert + c = \frac{1}{5}\ln\left(\frac{\tan\left(\frac{x}{2}\right)+2}{\frac{1}{2}-\tan\left(\frac{x}{2}\right)}\right) + c$ $= \frac{1}{5}\ln\left(\frac{2\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + c = \frac{1}{5}\ln\left(\frac{\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + \frac{1}{5}\ln 2 + c$ $= \frac{1}{5}\ln\left(\frac{\left(\tan\left(\frac{x}{2}\right)+2\right)}{1-2\tan\left(\frac{x}{2}\right)}\right) + k$	A1*	2.1
		(8 marks)	

Notes
B1: Uses the **correct** formulae to express $4\cos x - 3\sin x$ in terms of t B1(M1 on ePEN): Correct equation in terms of dx, dt and t – can be implied if seen as part of their substitution. M1: Makes a **complete** substitution to obtain an integral in terms of t only. Allow slips with the substitution of "dx" but must be dx = f(t)dt where f(t) ≠ 1. This mark is also available if the candidate makes errors when attempting to simplify $4\left(\frac{1-t^2}{1+t^2}\right)-3\left(\frac{2t}{1+t^2}\right)$ before attempting the substitution. A1: For obtaining a fully correct simplified integral with a constant in the numerator and a 3 term quadratic expression in the denominator. ("dt" not required) M1: Realises the need to express the integrand in terms of partial fractions in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.** A1: Correct integral in terms of partial fractions – allow any equivalent **correct** integral. ("dt" not required) A1: Fully correct integration in terms of t A1*: Correct solution with no errors including "+ k" (allow "+ c") and with the constant dealt with correctly if necessary. The denominator must also be dealt with correctly. E.g. if it appears as $2t - 1$ initially and becomes $1 - 2t$ without justification, this final mark should be withheld. **<u>Alternative for final 4 marks:</u>** M1: Realises the need to express the integrand in completed square form in order to attempt the integration. **Must have a 3 term quadratic expression in the denominator and a constant in the numerator.** A1: Correct integral with the square completed – allow any equivalent **correct** integral ("dt" not required) A1: Fully correct integration in terms of t A1*: Correct solution with no errors including "+ k" (allow "+ c") and with the constant dealt with correctly if necessary as shown in the scheme and with the denominator dealt with correctly if necessary. Note that it is acceptable for the "dt" to appear and disappear throughout the proof as long as the intention is clear.

8.

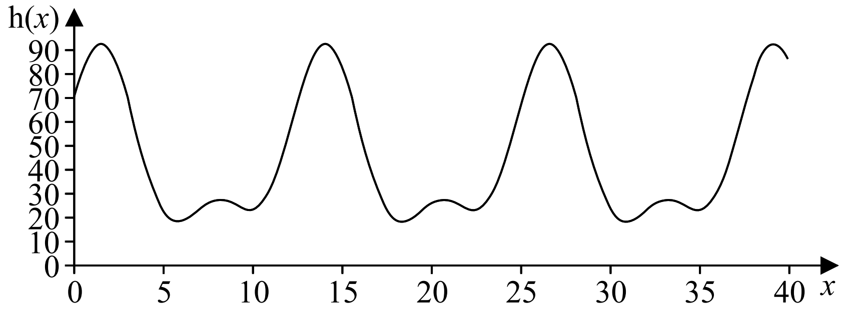


Figure 1

Figure 1 shows the graph of the function h(x) with equation

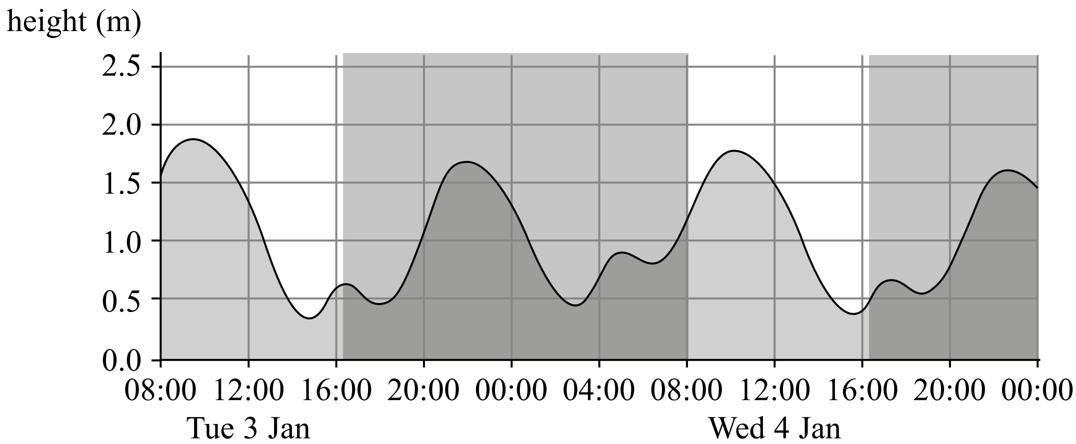
$$h(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right) \qquad x \in [0, 40]$$

(a) Show that

$$\frac{dh}{dx} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1 + t^2)^2}$$

where $t = \tan\left(\frac{x}{4}\right)$.

(6)



Source: [1]Data taken on 29th December 2016 from http://www.ukho.gov.uk/easytide/EasyTide

Figure 2

Figure 2 shows a graph of predicted tide heights, in metres, for Portland harbour from 08:00 on the 3rd January 2017 to the end of the 4th January 2017[1].

The graph of kh(x), where k is a constant and x is the number of hours after 08:00 on 3rd of January, can be used to model the predicted tide heights, in metres, for this period of time.

(b) (i) Suggest a value of k that could be used for the graph of kh(x) to form a suitable model.

(ii) Why may such a model be suitable to predict the times when the tide heights are at their peaks, but not to predict the heights of these peaks?

(3)

(c) Use Figure 2 and the result of part (a) to estimate, to the nearest minute, the time of the highest tide height on the 4th January 2017.

(6)

Question	Scheme	Marks	AOs
8(a)	$\mathrm{h}(x) = 45 + 15\sin x + 21\sin\left(\frac{x}{2}\right) + 25\cos\left(\frac{x}{2}\right)$		
	$\frac{\mathrm{dh}}{\mathrm{d}x} = 15\cos x + \frac{21}{2}\cos\left(\frac{x}{2}\right) - \frac{25}{2}\sin\left(\frac{x}{2}\right)$	M1	1.1b
	$\frac{\mathrm{dh}}{\mathrm{d}x} = ...+..\frac{1-t^2}{1+t^2} - ..\frac{2t}{1+t^2}$	M1	1.1a
	e.g. $\frac{\mathrm{d}h}{\mathrm{d}x} = ..\left(2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1\right) + ...$ **or** $\frac{\mathrm{d}h}{\mathrm{d}x} = ..\frac{1-\left(\frac{2t}{1-t^2}\right)^2}{1+\left(\frac{2t}{1-t^2}\right)^2} + ...$	M1	3.1a
	e.g. $\frac{\mathrm{d}h}{\mathrm{d}x} = 15\left(2\left(\frac{1-t^2}{1+t^2}\right)^2 - 1\right) + \frac{21}{2}\left(\frac{1-t^2}{1+t^2}\right) - \frac{25}{2}\left(\frac{2t}{1+t^2}\right)$	A1	1.1b
	$... = \frac{15[4(1-t^2)^2 - 2(1+t^2)^2] + 21(1-t^2)(1+t^2) - 50t(1+t^2)}{2(1+t^2)^2}$ x	M1	2.1
	$... = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2-6t-17)(9t^2+4t-3)}{2(1+t^2)^2}$ *	A1*	2.1
		(6)	
	8(a) Alternative $\mathrm{h}(x) = ... + 21\left(\frac{2t}{1+t^2}\right) + 25\left(\frac{1-t^2}{1+t^2}\right)$	M1	1.1a
	$= ... + 15\left[2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)\right] + ...$ **or** $= ... + 15\left(\frac{2\left(\frac{2t}{1-t^2}\right)}{1+\left(\frac{2t}{1-t^2}\right)^2}\right) + ...$	M1	2.1
	$\mathrm{h}(x) = 45 + \frac{15(4t(1-t^2)) + 42t(1+t^2) + 25(1-t^4)}{(1+t^2)^2}$	M1	1.1b
	$\mathrm{h}(x) = 45 - \frac{25t^4 + 18t^3 - 102t - 25}{(1+t^2)^2}$ or $\frac{20t^4 - 18t^3 + 90t^2 + 102t + 70}{(1+t^2)^2}$	A1	1.1b
	$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{('u')(1+t^2)^2 - ('u')(4t(1+t^2))}{(1+t^2)^4} \times \frac{1}{4}(1+t^2)$	M1	3.1a
	$... = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2-6t-17)(9t^2+4t-3)}{2(1+t^2)^2}$ *	A1*	2.1
		(6)	

Question	Scheme	Marks	AOs
8(b)(i)	Accept any value between $\frac{1}{40} = 0.025$ and $\frac{1}{60} \approx 0.167$ inclusive	B1	3.3
(ii)	Suitable for times since the graphs both oscillate bi-modally with about the same periodicity	B1	3.4
	Not suitable for predicting heights since the heights of the peaks vary over time, but the graph of h(x) has fixed peak height	B1	3.5b
		(3)	
8(c)	Solves at least one of the quadratics $t = \frac{6 \pm \sqrt{36 - 4\times1\times17}}{2} = 3 \pm \sqrt{26}$ or $t = \frac{-4 \pm \sqrt{16 - 4\times9\times(-3)}}{18} = \frac{-2 \pm \sqrt{31}}{9}$	**M1**	1.1b
	Finds corresponding x values, $x = 4\tan^{-1}(t)$ for at least one value of t from the $9t^2 + 4t - 3$ factor	**M1**	1.1b
	One correct value for these x e.g. $x =$ awrt -2.797 or $9.770, 1.510$	**A1**	1.1b
	Maximum peak height occurs at smallest positive value of x, from first graph, but the third of these peaks needed, So $t = 1.509\ldots + 8\pi = 26.642$ is the is the required time	**M1**	3.4
	$x = 26.642$ corresponds to 26 hours and 39 minutes (nearest minute) after 08:00 on 3rd January (Allow if a different greatest peak height used)	**M1**	3.4
	Time of greatest tide height is approximately 10:39 (am) (also allow 10:38 or 10:40)	**A1**	3.2a
		(6)	
		(15 marks)	

Notes:

(a)

M1: Differentiates h(x)

M1: Applies t-substitution to both $\left(\frac{x}{2}\right)$ terms with their coefficients

M1: Forms a correct expression in t for the cos x term, using double angle formula and t-substitution, or double 't'-substitution

A1: Fully correct expression in t for $\frac{\mathrm{d}h}{\mathrm{d}x}$

M1: Gets all terms over the correct common factor. Numerators must be appropriate for their terms

A1*: Achieves the correct answer via expression with correct quartic numerator before factorisation

Question 8 notes continued:

Alternative:

(a)

M1: Applies t-substitution to both $\left(\frac{x}{2}\right)$ terms

M1: Forms a correct expression in t for the $\sin x$ term, using double angle formula and t-substitution, or double 't'-substitution

M1: Gets all terms in t over the correct common factor. Numerators must be appropriate for their terms. May include the constant term too

A1: Fully correct expression in t for $\mathrm{h}(x)$

M1: Differentiates, using both chain rule and quotient rule with their 'u'

A1*: Achieves the correct answer via expression with correct quartic numerator before factorisation

Note: The individual terms may be differentiated before putting over a common denominator. In this case score the third M for differentiating with chain rule and quotient rule, then r return to the original scheme

(b)(i)

B1: Any value between $\frac{1}{40}$ (e.g. taking $\mathrm{h}(0)$ as reference point) **or** $\frac{1}{60}$ (taking lower peaks as reference)

NB: **Taking high peak as reference gives $\frac{1}{50}$**

(b)(ii)

B1: Should mention both the bimodal nature and periodicity for the actual data match the graph of h

B1: Mentions that the heights of peaks vary in each oscillation

(c)

M1: Solves (at least) one of the quadratic equations in the numerator

M1: Must be attempting to solve the quadratic factor from which the solution comes $9t^2 + 4t - 3$ and using $t = \tan\left(\frac{x}{4}\right)$ to find a corresponding value for x

A1: At least one correct x value from solving the requisite quadratic: awrt any of – 2.797, 1.510, 9.770, 14.076, 22.336, 26.642, 34.902 or 39.208

M1: Uses graph of h to pick out their $x = 26.642$ as the time corresponding to the third of the higher peaks, which is the highest of the peaks on 4th January on the tide height graph. As per scheme or allow if all times listed and correct one picked

M1: Finds the time for one of the values of t corresponding to the highest peaks. E.g. 1.5096...~ 09:31 (3rd January) or 14.076... ~ 22:05 (3rd January) or 26.642…~ 10:39 (4th January) or 39.208…~ 23:13 (4th January). (Only follow through on use of the smallest positive t solution $+\ 4k\pi$)

A1: Time of greatest tide height on 4th January is approximately 10:39. Also allow 10:38 or 10:40