OCR
Oxford Cambridge and RSA

Monday 3 June 2019 – Morning

A Level Further Mathematics A

Y540/01 Pure Core 1

Time allowed: 1 hour 30 minutes

A Level Further Mathematics A (H245)												
				Max Mark	a*	a	b	c	d	e	u
Y540	01	Pure Core 1	Raw	75	61	51	43	35	27	20	0
Y541	01	Pure Core 2	Raw	75	58	46	38	30	23	16	0
Y542	01	Statistics	Raw	75	63	54	47	41	35	29	0
Y543	01	Mechanics	Raw	75	52	42	36	30	24	18	0
Y544	01	Discrete Mathematics	Raw	75	51	45	38	31	24	18	0
Y545	01	Additional Pure Mathematics	Raw	75	56	47	39	31	24	17	0
H245		*Option Y540+Y541+Y542+Y543*	Overall	300	234	193	165	137	110	83	0
H245		*Option Y540+Y541+Y542+Y544*	Overall	300	233	196	167	139	111	83	0
H245		*Option Y540+Y541+Y542+Y545*	Overall	300	238	198	169	140	111	82	0
H245		*Option Y540+Y541+Y543+Y544*	Overall	300	222	184	156	128	100	72	0
H245		*Option Y540+Y541+Y543+Y545*	Overall	300	227	186	157	128	99	71	0
H245		*Option Y540+Y541+Y544+Y545*	Overall	300	226	189	159	129	100	71	0

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Give non-exact numerical answers correct to 3 significant figures unless a different degree of accuracy is specified in the question.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION
- The total mark for this paper is **75**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **16** pages. The Question Paper consists of **8** pages.

Answer **all** the questions.

1 **In this question you must show detailed reasoning.**

The quadratic equation $x^2 - 2x + 5 = 0$ has roots α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. **[1]**

(b) Hence find a quadratic equation with roots $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$. **[3]**

2 Indicate by shading on an Argand diagram the region

$\{z : |z| \leqslant |z-4|\} \cap \{z : |z-3-2\mathrm{i}| \leqslant 2\}$. **[3]**

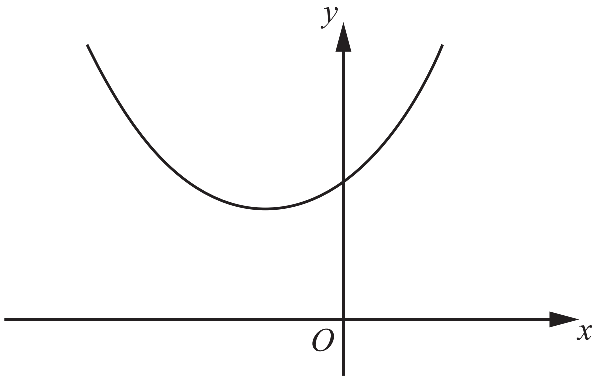
3 **In this question you must show detailed reasoning.**

You are given that $x = 2 + 5\mathrm{i}$ is a root of the equation $x^3 - 2x^2 + 21x + 58 = 0$.

Solve the equation. **[4]**

4 Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that $\sum_{r=1}^{10} r(3r-2) = 1045$. **[3]**

5 The diagram shows part of the curve $y = 5\cosh x + 3\sinh x$.



(a) Solve the equation $5\cosh x + 3\sinh x = 4$ giving your solution in exact form. **[4]**

(b) **In this question you must show detailed reasoning.**

Find $\int_{-1}^{1}(5\cosh x + 3\sinh x)\,\mathrm{d}x$ giving your answer in the form $a\mathrm{e} + \frac{b}{\mathrm{e}}$ where a and b are integers to be determined. **[3]**

6 You are given that $y = \tan^{-1}\sqrt{2x}$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. **[2]**

(b) Show that $\int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\sqrt{x}}{(x + 2x^2)}\,\mathrm{d}x = k\pi$ where k is a number to be determined in exact form. **[4]**

7 The function $\operatorname{sech} x$ is defined by $\operatorname{sech} x = \frac{1}{\cosh x}$.

(a) Show that $\operatorname{sech} x = \frac{2\mathrm{e}^x}{\mathrm{e}^{2x} + 1}$. **[2]**

(b) Using a suitable substitution, find $\int \operatorname{sech} x\,\mathrm{d}x$. **[4]**

8 The equation of a plane is $4x + 2y + z = 7$.
The point A has coordinates (9, 6, 1) and the point B is the reflection of A in the plane.

Find the coordinates of the point B. **[6]**

9 **In this question you must show detailed reasoning.**

You are given the complex number $\omega = \cos\frac{2}{5}\pi + \mathrm{i}\sin\frac{2}{5}\pi$ and the equation $z^5 = 1$.

(a) Show that ω is a root of the equation. **[2]**

(b) Write down the other four roots of the equation. **[1]**

(c) Show that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. **[2]**

(d) Hence show that $\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$. **[3]**

(e) Hence determine the value of $\cos\frac{2}{5}\pi$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers to be found. **[4]**

10 You are given the matrix $\mathbf{A}$ where $\mathbf{A} = \begin{pmatrix} a & 2 & 0 \\ 0 & a & 2 \\ 4 & 5 & 1 \end{pmatrix}$.

(a) Find, in terms of a, the determinant of $\mathbf{A}$, simplifying your answer. **[2]**

(b) Hence find the values of a for which $\mathbf{A}$ is singular. **[2]**

You are given the following equations which are to be solved simultaneously.

$$\begin{aligned} ax + 2y \phantom{{}+ 2z} &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

(c) For each of the values of a found in part **(b)** determine whether the equations have

- a unique solution, which should be found, or
- an infinite set of solutions or
- no solution. **[7]**

11 A particle is suspended in a resistive medium from one end of a light spring. The other end of the spring is attached to a point which is made to oscillate in a vertical line.

The displacement of the particle may be modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10\sin t$$

where x is the displacement of the particle below the equilibrium position at time t.

When $t = 0$ the particle is stationary and its displacement is 2.

(a) Find the particular solution of the differential equation. **[11]**

(b) Write down an approximate equation for the displacement when t is large. **[2]**

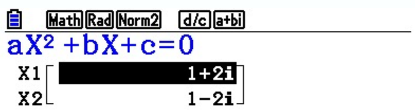
END OF QUESTION PAPER

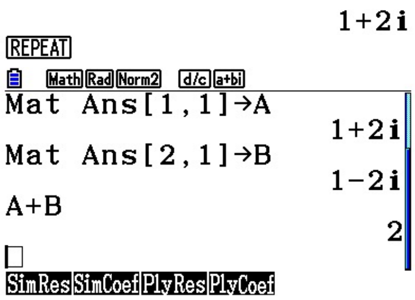
1 In this question you must show detailed reasoning.

The quadratic equation $x^2 - 2x + 5 = 0$ has roots α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. **[1]**

(b) Hence find a quadratic equation with roots $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$. **[3]**





(a)

$\alpha + \beta = 2$

$\alpha\beta = 5$

(b) $\alpha + \frac{1}{\beta} + \beta + \frac{1}{\alpha} = S$

$= \alpha + \beta + \frac{1}{\alpha} + \frac{1}{\beta}$

$= \alpha + \beta + \frac{\beta + \alpha}{\alpha\beta}$

$= 2 + \frac{2}{5} = \frac{10+2}{5} = \frac{12}{5}$ //

$P = (\alpha + \frac{1}{\beta})(\beta + \frac{1}{\alpha})$

$= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta}$

$= 5 + 1 + 1 + \frac{1}{5} = 7 + \frac{1}{5} = \frac{36}{5}$ //

$x^2 - \frac{12}{5}x + \frac{36}{5} = 0$

$5x^2 - 12x + 36 = 0$ //

$\boxed{\frac{6}{5} + \frac{12}{5}i}$

2 Indicate by shading on an Argand diagram the region

$\{z : |z| \leqslant |z-4|\} \cap \{z : |z-3-2\mathrm{i}| \leqslant 2\}$.

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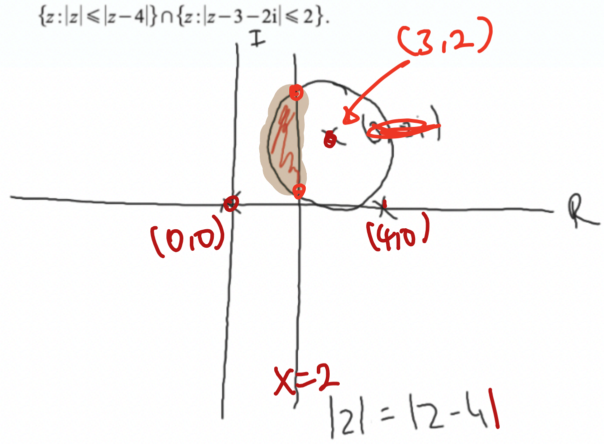


$|z| \leq |z-4|$

$|x+iy| \leq |x+iy-4|$

$\sqrt{x^2+y^2} \leq \sqrt{(x-4)^2+y^2}$

$x^2+y^2 \leq x^2+y^2-8x+16$

$8x \leq 16$

$x \leq 2$

$|x+iy-3-2i| \leq 2$

$\sqrt{(x-3)^2+(y-2)^2} \leq 2$

$(x-3)^2+(y-2)^2 \leq 2^2$

3 **In this question you must show detailed reasoning.**

$\alpha+\beta+r$

You are given that $x = 2+5\mathrm{i}$ is a root of the equation $x^3 - 2x^2 + 21x + 58 = 0$.

Solve the equation. **[4]**

$$(x-(2+5i))(x-(2-5i))(x+r)=0$$

$$(x^2-2x-\cancel{5ix}-2x+\cancel{5ix}+29)(x+r)=0$$

$$\underbrace{(x^2-4x+29)}(x+2)=0$$

$x=-2$, $2+5i$, $2-5i$

$\alpha = 2+5i$

$\beta = 2-5i$ (conjugates)

$x^2 - Sx + P = 0$

$x^2 - 4x + 29 = 0$

$\alpha+\beta = 4$

$\alpha\beta = (2+5i)(2-5i) = 2^2+5^2 = 4+25 = 29$

Way1 $(x^2-4x+29)(x+\lambda)=0$

$(29)(\lambda)=58$

$\lambda = 2$

$$\boxed{\begin{array}{l} x=2+5i \\ x=2-5i \\ x=-2 \end{array}}$$

Way2 $\alpha+\beta+r=2$

$2+5i+2-5i+r=2$

$4+r=2$

$r=-2$

Series

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

4 Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, show that $\sum_{r=1}^{10} r(3r-2) = 1045$. $a=1 \quad l=n$ **[3]**

$\sum_{1}^{n} r = \frac{1}{2}(n)(1+n)$

$\sum_{1}^{10} (r(3r-2))$

$\sum_{1}^{10} (3r^2 - 2r)$

$= 3\left(\frac{1}{6}n(n+1)(2n+1)\right) - 2\left(\frac{1}{2}n(n+1)\right)$

$= \frac{1}{2}n(n+1)(2n+1) - n(n+1) =$

$= \frac{1}{2} \times 10(11)(21) - 10(11)$

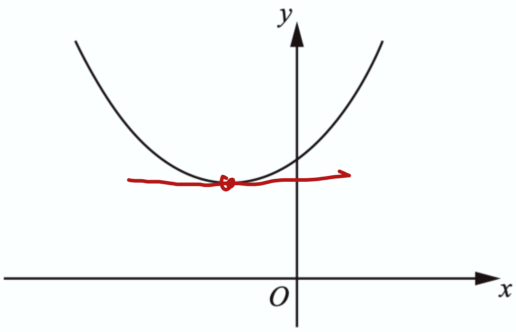
$1155 - 110 = \underline{\underline{1045}}$ ✓

Pure Mat

Arithmetic series

$S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a+(n-1)d\}$

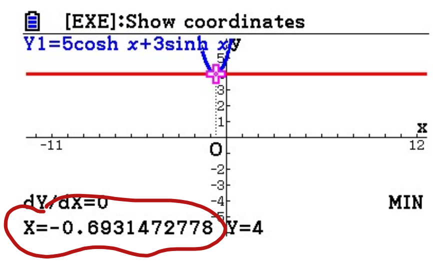
5 The diagram shows part of the curve $y = 5\cosh x + 3\sinh x$.

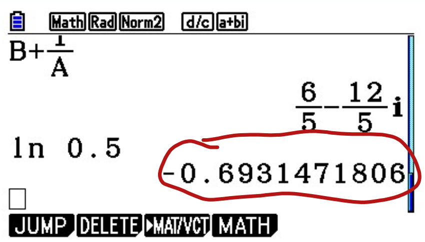


(a) Solve the equation $5\cosh x + 3\sinh x = 4$ giving your solution in exact form. **[4]**

(b) **In this question you must show detailed reasoning.**

Find $\int_{-1}^{1}(5\cosh x + 3\sinh x)\mathrm{d}x$ giving your answer in the form $a\mathrm{e} + \frac{b}{\mathrm{e}}$ where a and b are integers to be determined. **[3]**





5) a) $5\left(\frac{e^{x}+e^{-x}}{2}\right)+3\left(\frac{e^{x}-e^{-x}}{2}\right)=4$

$\frac{8}{2}e^{x}+\frac{2}{2}e^{-x}=4$

$4e^{x}+e^{-x}=4$

$4e^{2x}+1=4e^{x}$

$4e^{2x}-4e^{x}+1=0$

$e^{x}=0.5$ (repeated)

$x=\ln\frac{1}{2}$ ✓✓

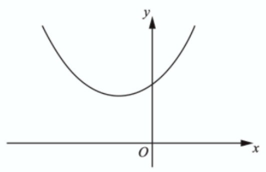
Math Rad Norm2 d/c a+bi

$\int_{-1}^{1} 5\cosh x + 3\sinh x\,dx$

11.75201194

sinh cosh tanh sinh⁻¹ cosh⁻¹ tanh⁻¹

5 The diagram shows part of the curve $y = 5\cosh x + 3\sinh x$.



$\cosh x = \frac{e^x + e^{-x}}{2}$

(a) Solve the equation $5\cosh x + 3\sinh x = 4$ giving your solution in exact form. **[4]**

(b) In this question you must show detailed reasoning.

Find $\int_{-1}^{1}(5\cosh x + 3\sinh x)\,dx$ giving your answer in the form $ae + \frac{b}{e}$ where a and b are integers to be determined. **[3]**

5) a)

$5\left(\frac{e^x + e^{-x}}{2}\right) + 3\left(\frac{e^x - e^{-x}}{2}\right) = 4$

$\frac{8}{2}e^x + \frac{2}{2}e^{-x} = 4$

$4e^x + e^{-x} = 4$

$4e^{2x} + 1 = 4e^x$

$4e^{2x} - 4e^x + 1 = 0$

$e^x = 0.5$

$x = \ln\frac{1}{2}$

b)

$\int 5\cosh x + 3\sinh x\,dx$

$\left[5\sinh x + 3\cosh x\right]_{-1}^{1}$

$\left[5\sinh 1 + 3\cosh 1\right] - \left[5\sinh -1 + 3\cosh -1\right]$

$= \left[5\left(\frac{e^1 - e^{-1}}{2}\right) + 3\left(\frac{e^1 + e^{-1}}{2}\right)\right] - \left[5\left(\frac{e^{-1} - e^{+1}}{2}\right) + 3\left(\frac{e^{-1} + e^{+1}}{2}\right)\right]$

$5e^1 - 5e^{-1}$

6 You are given that $y = \tan^{-1}\sqrt{2x}$.

(a) Find $\frac{dy}{dx}$. **[2]**

(b) Show that $\int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\sqrt{x}}{(x+2x^2)} dx = k\pi$ where k is a number to be determined in exact form. **[4]**

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tan
sec
sec

$\sqrt{2}\ x^{\frac{1}{2}}$

$\{\sin^2 y + \cos^2 y = 1$

$\tan^2 y + 1 = \sec^2 y\}$

$y = \tan^{-1}\sqrt{2x}$

$\tan y = \sqrt{2x}$

$\sec^2 y \frac{dy}{dx} = \frac{\sqrt{2}}{2} x^{-\frac{1}{2}}$

$(\tan^2 y + 1)\frac{dy}{dx} = \frac{\sqrt{2}}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{\sqrt{2}}{(\tan^2 y+1)(2\sqrt{x})} = \frac{\sqrt{2}}{(2x+1)(2\sqrt{x})}$

Note $\tan^2 y + 1$

$= (\sqrt{2x})^2 + 1$

$= 2x + 1$

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(a) Find $\frac{dy}{dx}$. [2]

(b) Show that $\int_{\frac{1}{6}}^{\frac{1}{2}} \frac{\sqrt{x}}{(x+2x^2)}\,dx = k\pi$ where k is a number to be determined in exact form. [4]

tan
sec
sec

$\sqrt{2}\ x^{\frac{1}{2}}$

$\sin^2 y + \cos^2 y = 1$

$\tan^2 y + 1 = \sec^2 y$

$x^{\frac{1}{2}} \cdot x^{-1} = x^{-\frac{1}{2}}$

$y = \tan^{-1}\sqrt{2x}$

$\tan y = \sqrt{2x}$

$\sec^2 y \frac{dy}{dx} = \frac{\sqrt{2}}{2} x^{-\frac{1}{2}}$

$(\tan^2 y + 1)\frac{dy}{dx} = \frac{\sqrt{2}}{2\sqrt{x}}$

$\frac{dy}{dx} = \frac{\sqrt{2}}{(\tan^2 y + 1)(2\sqrt{x})} = \frac{\sqrt{2}}{(2x+1)(2\sqrt{x})}$

Note $\tan^2 y + 1$
$= (\sqrt{2x})^2 + 1$
$= 2x + 1$

$\frac{\sqrt{x}}{(1+2x)(x)} = \frac{1}{(1+2x)\sqrt{x}}$

$\frac{2}{\sqrt{2}} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{1}{\sqrt{x}(1+2x)} \frac{\sqrt{2}}{2}\,dx$

$= \frac{2}{\sqrt{2}} \tan^{-1}\left[\sqrt{2x}\right]_{\frac{1}{6}}^{\frac{1}{2}}$

$\frac{2}{\sqrt{2}} \int_{\frac{1}{6}}^{\frac{1}{2}} \frac{1}{\sqrt{x}(1+2x)} \frac{\sqrt{2}}{2}\,dx$

$= \frac{2}{\sqrt{2}} \tan^{-1}\left[\sqrt{2x}\right]_{\frac{1}{6}}^{\frac{1}{2}}$

$= \left[\frac{2}{\sqrt{2}} \tan^{-1}\sqrt{1}\right] - \left[\frac{2}{\sqrt{2}} \tan^{-1}\sqrt{\frac{1}{3}}\right]$

$= \left(\frac{2}{\sqrt{2}} \times \frac{\pi}{4}\right) - \left(\frac{2}{\sqrt{2}} \times \frac{\pi}{6}\right)$

$= \frac{2}{12\sqrt{2}}\pi$

7 The function $\operatorname{sech} x$ is defined by $\operatorname{sech} x = \dfrac{1}{\cosh x}$.

(a) Show that $\operatorname{sech} x = \dfrac{2e^x}{e^{2x}+1}$. **[2]**

(b) Using a suitable substitution, find $\int \operatorname{sech} x \, dx$. **[4]**

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(a) Show that $\operatorname{sech} x = \dfrac{2e^x}{e^{2x}+1}$. **[2]**

(b) Using a suitable substitution, find $\int \operatorname{sech} x \, dx$. **[4]**

7) a)

$$= \frac{1}{\cosh x}$$

$$= \frac{2}{e^x + e^{-x}} \times \frac{e^x}{e^x}$$

$$= \frac{2e^x}{e^{2x}+1}$$

b) $e^x = u$

$$\frac{du}{dx} = e^x = u$$

$$\frac{du}{u} = dx$$

$$\int \operatorname{sech} x \, dx$$

$$= \int \frac{2e^x}{e^{2x}+1} \, dx$$

$$= \int \frac{2u}{u^2+1} \times \frac{1}{u} \, du$$

$$= 2\int \frac{1}{u^2+1} \, du$$

$$= 2\tan^{-1} u$$

$$2\tan^{-1} e^x + c$$

8 The equation of a plane is $4x+2y+z=7$.
The point A has coordinates (9, 6, 1) and the point B is the reflection of A in the plane.

Find the coordinates of the point B. **[6]**

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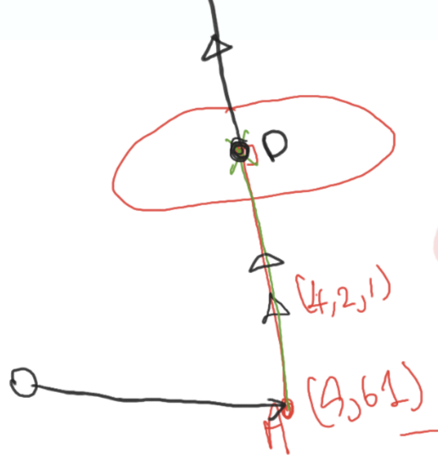


$r \cdot n = a \cdot n$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = 7$

$r = \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

$r \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = 7$

$\begin{pmatrix} 9+4\lambda \\ 6+2\lambda \\ 1+\lambda \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = 7$

$4(9+4\lambda)+2(6+2\lambda)+(1+\lambda)=7$

$36+16\lambda+12+4\lambda+1+\lambda=7$

$21\lambda+49=7$

$\lambda=-2$

$OB = OA + 2(-2)\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix} - 4\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \\ -3 \end{pmatrix}$ //

9 **In this question you must show detailed reasoning.**

You are given the complex number $\omega = \cos\frac{2}{5}\pi + \mathrm{i}\sin\frac{2}{5}\pi$ and the equation $z^5 = 1$.

(a) Show that ω is a root of the equation. **[2]**

(b) Write down the other four roots of the equation. $-\pi < z < \pi$ **[1]**

(c) Show that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. **[2]**

(d) Hence show that $\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$. **[3]**

(e) Hence determine the value of $\cos\frac{2}{5}\pi$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers to be found. **[4]**

(a)

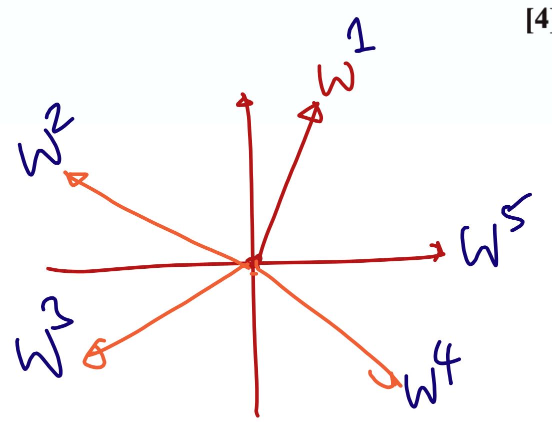
$W = \cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi$

$W^5 = \left(\cos\frac{2}{5}\pi + i\sin\frac{2}{5}\pi\right)^5$

$= \cos 2\pi + i\sin 2\pi$

$= 1 + 0i$

$= 1$ $\Rightarrow$ root



(b)

$z_1 = e^{\frac{2\pi}{5}i}$ $z_2 = e^{\frac{4\pi}{5}i}$ $z_3 = e^{-\frac{4\pi}{5}i}$ $z_4 = e^{-\frac{2\pi}{5}i}$

$z_5 = 1$

(c) $z_1 = W$ $z_2 = W^2$ $z_3 = W^3$ $z_4 = W^4$ $z_5 = W^5$

$a = w \quad r = w \qquad \dfrac{a(1-r^4)}{1-r}$

$w = e^{\frac{2\pi}{5}i} \qquad \dfrac{w(1-w^4)}{1-w}$

$w + w^2 + w^3 + w^4 = \dfrac{e^{\frac{2\pi}{5}i}\left(1 - e^{-\frac{2\pi}{5}i}\right)}{1 - e^{\frac{2\pi}{5}i}}$

$= \dfrac{e^{\frac{2\pi}{5}i} - e^{0}}{1 - e^{\frac{2\pi}{5}i}}$

$= \dfrac{e^{\frac{2\pi}{5}i} - 1}{1 - e^{\frac{2\pi}{5}i}}$

$= -1 \Rightarrow$ as required //

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You are given the complex number $\omega = \cos\frac{2}{5}\pi + \mathrm{i}\sin\frac{2}{5}\pi$ and the equation $z^5 = 1$.

(a) Show that ω is a root of the equation. **[2]**

(b) Write down the other four roots of the equation. $-\pi < z < \pi$ **[1]**

(c) Show that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. **[2]**

(d) Hence show that $\left(\omega + \dfrac{1}{\omega}\right)^2 + \left(\omega + \dfrac{1}{\omega}\right) - 1 = 0$. **[3]**

(e) Hence determine the value of $\cos\frac{2}{5}\pi$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers to be found. **[4]**

$z_3 = e^{\frac{6\pi}{5}i} \quad z_4 = e^{\frac{8\pi}{5}i}$

(b) $z_1 = e^{\frac{2\pi}{5}i}$ $z_2 = e^{\frac{4\pi}{5}i}$ $z_3 = e^{-\frac{4\pi}{5}i}$ $z_4 = e^{-\frac{2\pi}{5}i}$

$z_5 = 1$

(c) $z_1 = w$ $z_2 = w^2$ $z_3 = w^3$ $z_4 = w^4$ $z_5 = w^5$

(d)

$$\begin{aligned}
LHS &= \left(e^{\frac{2\pi}{5}i} + e^{-\frac{2\pi}{5}i}\right)^2 + \left(e^{\frac{2\pi}{5}i} + e^{-\frac{2\pi}{5}i}\right) - 1 \\
&= (w + w^{-1})^2 + (w + w^{-1}) - 1 \\
&= (w^2 + w^{-2} + 2) + w + w^{-1} - 1 \\
&= w^{-2} + w^{-1} + w + w^2 + 1 \\
&= -1 + 1 \\
&= 0
\end{aligned}$$

(c) Show that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. [2]

(d) Hence show that $\left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega + \frac{1}{\omega}\right) - 1 = 0$. [3]

$w = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$$\frac{1}{w} = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^{-1}$$

$$= \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5}$$

$$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

(e) $w + \frac{1}{w} = 2\cos\frac{2\pi}{5}$

$$\left(2\cos\frac{2\pi}{5}\right)^2 + \left(2\cos\frac{2\pi}{5}\right) - 1 = 0$$

$$4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$$

$$\cos\frac{2\pi}{5} = \frac{-1 \pm \sqrt{5}}{4}$$

(e) Hence determine the value of $\cos\frac{2}{5}\pi$ in the form $a + b\sqrt{c}$ where a, b and c are rational numbers to be found. **[4]**

Math Rad Norm2 d/c a+bi

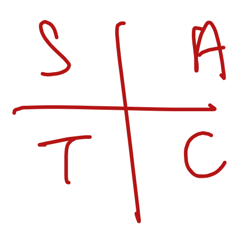
$aX^2 + bX + c = 0$

X1 0.309

X2 -0.809

$$\frac{-1+\sqrt{5}}{4}$$

REPEAT



$\cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$ only.

10 You are given the matrix **A** where $\mathbf{A} = \begin{pmatrix} a & 2 & 0 \\ 0 & a & 2 \\ 4 & 5 & 1 \end{pmatrix}$.

(a) Find, in terms of a, the determinant of **A**, simplifying your answer. **[2]**

(b) Hence find the values of a for which **A** is singular. **[2]**

You are given the following equations which are to be solved simultaneously.

$$\begin{aligned} ax + 2y &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

(c) For each of the values of a found in part **(b)** determine whether the equations have

- a unique solution, which should be found, or
- an infinite set of solutions or
- no solution.

[7]

10) a)

$\det A = a\left[\det\begin{pmatrix} a & 2 \\ 5 & 1 \end{pmatrix}\right] - 2\left[\det\begin{pmatrix} 0 & 2 \\ 4 & 1 \end{pmatrix}\right]$

$+ 0 \left[\det\begin{pmatrix} 0 & a \\ 4 & 5 \end{pmatrix}\right]$

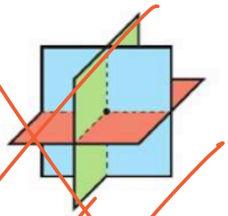
$= a\,[a-10) \;-2(-8) + 0$

$= \underline{a^2 - 10a + 16}$

b) . $a^2 - 10a + 16 = 0$

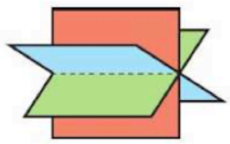
$a = 8$

$a = 2$



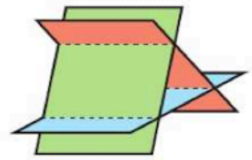
~~The planes meet at a **point**. The system of equations is **consistent** and has **one solution** represented by this point. This is the only case when the corresponding matrix is **non-singular**.~~

1



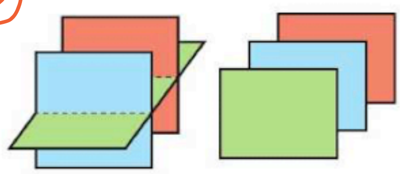
The planes form a **sheaf**. The system of equations is **consistent** and has **infinitely many solutions** represented by the line of intersection of the three planes.

2



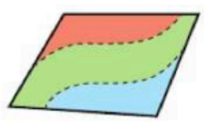
The planes form a **prism**. The system of equations is **inconsistent** and has **no solutions**.

3



Two or more of the planes are parallel and non-identical. The system of equations is **inconsistent** and has **no solutions**.

4



All three equations represent the same plane. In this case the system of equations is **consistent** and has **infinitely many solutions**.

Hint If one row of the corresponding matrix is a **linear multiple** of another row, then these two rows will represent parallel planes.

$$\begin{aligned} ax + 2y \quad &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

(c) For each of the values of a found in part **(b)** determine whether the equations have

- a unique solution, which should be found, or
- an infinite set of solutions or
- no solution.

[7]

$a = 8$

$$8x + 2y = 6 \quad -1$$

$$8y + 2z = 8 \quad -2$$

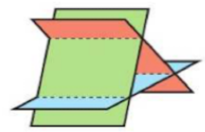
$$4x + 5y + z = 16 \quad -3$$

$$4x + y = 3 \quad ① \times \tfrac{1}{2}$$
$$4x + 5y + z = 16 \quad ③$$

$$4y + z = 13 \qquad ③ - \tfrac{1}{2}①$$
$$8y + 2z = 8$$

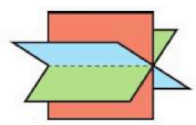
inconsistent

$a = 8 \Rightarrow ②$



The planes form a **prism**. The system of equations is **inconsistent** and has **no solutions**.

$a=2$ ①



The planes form a **sheaf**. The system of equations is **consistent** and has **infinitely many solutions** represented by the line of intersection of the three planes.

$a = 2$

$$\begin{aligned} ax + 2y \phantom{{}+ 2z} &= 6 \\ ay + 2z &= 8 \\ 4x + 5y + z &= 16 \end{aligned}$$

$$2x + 2y = 6 \quad ①$$
$$2y + 2z = 8 \quad ②$$
$$4x + 5y + z = 16 \quad ③$$

Math Rad Norm2 d/c a+bi

$a_n X + b_n Y + C_n Z = d_n$

Infinitely
Many Solutions
X=−1+Z
Y=4−Z
Z=Z

REPEAT

② − ①:

$$-2x + 2z = 2$$

$5 \times ② - 2 \times ③$

$$\cancel{10y} + 10z - 8x - \cancel{10y} - 2z = 5(8) - 2(16)$$

$-8x+8z=8$

consistent

$4(-2x+2z=2) \Rightarrow -8x+8z=8$

a) $CF + PI$

CF: $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5 = 0$

$e^{-1t}\left[A\cos 2\cancel{t} + B\sin 2t\right] = x$

PI: let $x = w\sin t + p\cos t$

$\frac{dx}{dt} = w\cos t - p\sin t$

$\frac{d^2x}{dt^2} = -w\sin t - p\cos t$

$(-w\sin t - p\cos t) + 2(w\cos t - p\sin t) + 5(w\sin t + p\cos t) = 10\sin t$

$-w - 2p + 5w = 10$

$-p + 2w + 5p = 0$

$4w - 2p = 10$

$2w + 4p = 0$

$w = 2$

$p = -1$ //

11 A particle is suspended in a resistive medium from one end of a light spring. The other end of the spring is attached to a point which is made to oscillate in a vertical line.

The displacement of the particle may be modelled by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10\sin t$$

where x is the displacement of the particle below the equilibrium position at time t.

When $t = 0$ the particle is stationary and its displacement is 2.

(a) Find the particular solution of the differential equation. **[11]**

(b) Write down an approximate equation for the displacement when t is large. **[2]**

$e^{-1}[A\cos$

CF: $m^2 + 2m + 5 = 0$

$m = -1 \pm 2i$

$x = e^{-t}(A\cos 2t + B\sin 2t)$

$GS = CF + PI = e^{-t}(A\cos 2t + B\sin 2t) + 2\sin t + -1\cos t = x$

When $t = 0$ the particle is stationary and its displacement is 2.

$t = 0$

$\frac{dx}{dt} = 0$

$x = 2$

$2 = 1(A + 0) - 1$

$2 = A - 1$

$A = 3$

$$\frac{dx}{dt} = e^{-t}\left[-2A\cancel{\sin 2t} + 2B\cos 2t\right] + \left[A\cos 2t + B\cancel{\sin 2t}\right] - e^{-t} + 2\cos t + \cancel{\sin t}$$

$0 = 1(2B) - 1(A) + 2$

$-A + 2B + 2 = 0$

$2B = A - 2$

$2B = 1$

$B = 1/2$

$$e^{-t}(A\cos 2t + B\sin 2t) + 2\sin t + -1\cos t = x$$

(a) Find the particular solution of the differential equation. **[11]**

(b) Write down an approximate equation for the displacement when t is large. **[2]**

a

$$x = 3e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$$
$$+ 2\sin t - \cos t$$

$t \to \infty \qquad x = e^{-t} \to 0$

$$x \Rightarrow 2\sin t - \cos t$$