Further Mathematics A

Proof by Mathematical Induction (OCR)

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Please note that you may see slight differences between this paper and the original.

Candidates answer on the Question paper.

OCR supplied materials:
Additional resources may be supplied with this paper.

Other materials required:
- Pencil
- Ruler (cm/mm)

Duration: Not set

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the boxes above. Please write clearly and in capital letters.
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions, unless your teacher tells you otherwise.
- Read each question carefully. Make sure you know what you have to do before starting your answer.
- Where space is provided below the question, please write your answer there.
- You may use additional paper, or a specific Answer sheet if one is provided, but you must clearly show your candidate number, centre number and question number(s).

INFORMATION FOR CANDIDATES

- The quality of written communication is assessed in questions marked with either a pencil or an asterisk. In History and Geography a *Quality of extended response* question is marked with an asterisk, while a pencil is used for questions in which *Spelling, punctuation and grammar and the use of specialist terminology* is assessed.
- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- The total number of marks for this paper is **96**.
- The total number of marks may take into account some 'either/or' question choices.

1 The sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = 5^n + 2^{n-1}$.

(i) Find u_1, u_2 and u_3.

[2]

(ii) Hence suggest a positive integer, other than 1, which divides exactly into every term of the sequence.

[1]

(iii) By considering $u_{n+1} + u_n$, prove by induction that your suggestion in part **(ii)** is correct.

[5]

2 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Prove by induction that, for $n \geq 1$,

$$\mathbf{M}^n = \begin{pmatrix} 2^n & 2^{n+1} - 2 \\ 0 & 1 \end{pmatrix}.$$

[6]

3 Prove by induction that, for n ☐ 1, $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$.

[5]

4 Prove that $n! > 2^n$ for $n \geq 4$. **[5]**

5 A sequence is defined by $a_1 = 6$ and $a_{n+1} = 5a_n$ for $n \geq 1$.

Prove by induction that for all integers $n \geq 1$, $a_n = \frac{11 \times 5^{n-1} + 1}{2}$. **[5]**

Three matrices, **A**, **B** and *C*, are given by $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ a & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 5 & 0 \\ -2 & 2 \end{pmatrix}$ where *a* is a constant.

(a) Using **A**, **B** and **C** in that order demonstrate explicitly the associativity property of matrix multiplication. **[4]**

(b) Use A and C to disprove by counterexample the proposition 'Matrix multiplication is commutative'. **[2]**

For a certain value of *a*, $\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$.

(c) Find

- *y* in terms of *x*,
- the value of *a*. **[3]**

7 Prove by induction that, for all positive integers n, $7^n + 3^{n-1}$ is a multiple of 4. **[5]**

8

M is the matrix $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$.

Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & 3(2^{n+1}-2) \\ 0 & 2^n \end{pmatrix}$, for any positive integer *n*. **[6]**

9 Prove by induction that, for all positive integers *n*,

$$\sum_{r=1}^{n} \frac{5-4r}{5^r} = \frac{n}{5^n}.$$

[5]

10 Prove by induction that $n! \geq 6n$ for $n \geq 4$. **[5]**

11

Prove by induction that, for $n \geqslant 1, \sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{2n+1}$. **[5]**

12 **In this question you must show detailed reasoning.**

A sequence of vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, … is defined by

- $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

- $\mathbf{a}_{n+1} = (\mathbf{a}_n \times \mathbf{b}) \times \mathbf{b}$, for integers $n \geq 1$, where $\mathbf{b}$ is the vector $\frac{1}{4}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$.

(a) Prove by induction that $\mathbf{a}_n = \left(-\frac{7}{8}\right)^{n-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. for all integers $n \geq 1$. **[8]**

(b) Use an algebraic method to find the smallest value of n such that $|\mathbf{a}_n| < 0.001$. **[3]**

13 Prove by mathematical induction that, for all integers $n \geq 1$, $n^5 - n$ is divisible by 5. **[4]**

14 The 2×2 matrix **A** represents a transformation T which has the following properties.

- The image of the point (0, 1) is the point (3, 4).
- An object shape whose area is 7 is transformed to an image shape whose area is 35.
- T has a line of invariant points.

(a) Find a possible matrix for **A**. **[8]**

The transformation S is represented by the matrix **B** where $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$.

(b) Find the equation of the line of invariant points of S. **[2]**

(c) Show that any line of the form $y = x + c$ is an invariant line of S. **[3]**

15 The sequence u_1, u_2, u_3, … is defined by

$$u_1 = 5 \text{ and } u_{n+1} = 3u_n + 2 \text{ for } n \;☐\; 1.$$

Prove by induction that $u_n = 2 \times 3^n - 1$.

[4]

END OF QUESTION PAPER

Question			Answer/Indicative content	Marks	Part marks and guidance	
1		i	6 27	B1	Obtain correct values	
		i	129	B1	Obtain 3rd correct value **Examiner's Comments** Most candidates found the first 3 terms correctly and deduced a correct value for (ii) from their values.	
		ii	3	B1ft	State a correct value	
		iii	$5^{n+1}+2^n$	B1	Correct expression for u_{n+1} seen	Any letter, usually k or n
		iii		M1	Attempt to factorise u_{n+1} + u_n	Must deal with powers of 5 and 2
		iii		A1	Obtain correct simplified answer	e. g. $6 \times 5^n + 3 \times 2^{n-1}$
		iii		A1	Clear explanation why u_{n+1} is divisible by 3	Not u_{n+1} + u_n divisible by 3
		iii		B1	Clear statement of induction process **Examiner's Comments** Some candidates tried to prove directly that u_{n+1} has a divisor of 3, with limited success. Those who showed that u_{n+1} + u_n has a divisor of 3, then failed to explain clearly how this led to u_{n+1} having a divisor of 3.	Provided other 4 marks earned
			Total	8		
2				B1	Establish result true for n = 1 or n = 2	
				M1	Multiply **M** and $\mathbf{M}^k$, either order	
			$2\left(2^{k+1}-2\right)+2$ or $2^{k+1}+2^{k+1}-2$	A1	Obtain correct element	

Question			Answer/Indicative content	Marks	Part marks and guidance	
				A1	Obtain other 3 correct elements	
				A1	Obtain $2^{k+2} - 2$ convincingly	
				B1	Specific statement of induction conclusion, provided 5/5 earned so far and verified for $n = 1$ **Examiner's Comments** This was one of the less well-attempted questions. A significant minority thought that the induction step required the addition of $\mathbf{M}^k$ and **M**. Many failed to show sufficient working either in establishing the validity when $n = 1$, or in obtaining the elements in $\mathbf{M}^{k+1}$. Centres should remind candidates of this, as well as the need for a clear statement of the induction conclusion.	
			Total	6		

Question			Answer/Indicative content	Marks	Part marks and guidance	
3				B1	Show sufficient working to verify result true when $n = 1$	
			$k(k + 1)^2 + (k + 1)(3k + 4)$	M1*	Add next term in series	
				DM1	Attempt to factorise their expression	
			$(k + 1)(k + 2)^2$	A1	Sufficient working to obtain this correct answer	
				B1	Clear statement of induction process, provided previous 4 marks earned **Examiner's Comments** This was not done particularly well. Most could establish the truth of the result for $n = 1$, but then did not add on correct $(k + 1)$th term, or omitted brackets and so obtained an expression that could not be simplified. Those who had a correct expression often showed insufficient working to justify the truth for $k + 1$. Many lost the final mark by not giving a clear statement of how induction works.	
			Total	5		

Question			Answer/Indicative content	Marks	Part marks and guidance		
4			Let $n = 4$, then $4! = 24$ and $2^4 = 16$ so $4! > 2^4$ Assume true for $n = r$ $r! > 2^r$ for $r \geq 4$ Then for $n = r + 1$ $(r + 1)! = (r + 1) \times r! > (r + 1) \times 2^r$ by assumption Since $r + 1 > 2$, $(r + 1) \times 2^r > 2 \times 2^r = 2^{r+1}$ so $(r + 1)! > 2^{r+1}$ If true for r then true for $r + 1$. Hence, given basis case, the statement is true for all positive integers.	B1(AO2.1) M1(AO2.1) M1(AO1.1) E1(AO2.2a) E1(AO2.4) [5]	Basis case for proof by induction Assumption Add next statement Sufficient working to establish true for $r + 1$ Clear conclusion for induction process	Must state that $r + 1 > 2$ oe A *formal* proof is required for full marks Accept other *complete* methods	
			Total	5			

Question	Answer/Indicative content	Marks	Part marks and guidance
5	Base case, $n = 1$,	B1 (AO2.1)	
	$a_1 = \dfrac{11 \times 5^0 + 1}{2} = 6$ Assume true for $n = k$	M1 (AO2.1)	Statement of inductive hypothesis
	$\Rightarrow a_k = \dfrac{11 \times 5^{k-1} + 1}{2}$	M1 (AO2.2a)	Use of both recurrence relation for a_{k+1} in terms of a_k and also inductive hypothesis in attempt to derive required form for a_{k+1}
	The $a_{k+1} = 5a_k - 2 = 5\left(\dfrac{11 \times 5^{k-1} + 1}{2}\right) - 2$ $= \left(\dfrac{11 \times 5^k + 5}{2}\right) - 2 = \left(\dfrac{11 \times 5^k + 1 + 4}{2}\right) - 2$ n $= \left(\dfrac{11 \times 5^k + 1}{2}\right) + 2 - 2 = \dfrac{11 \times 5^{(k+1)-1} + 1}{2}$	A1 (AO2.5)	
	So if true for $n = k$ then true also for $n = k + 1$.	E1 (AO2.4)	
	But since it is true for $n = 1$ then it must be true for all integers $n \geq 1$.	[5]	
	Total	5	

Question			Answer/Indicative content	Marks	Part marks and guidance		
6		a	$\mathbf{AB}=\begin{pmatrix}1 & 2\\ a & -1\end{pmatrix}\begin{pmatrix}2 & -1\\ 4 & 1\end{pmatrix}=\begin{pmatrix}10 & 1\\ 2a-4 & -a-1\end{pmatrix}$ $(\mathbf{AB})\mathbf{C}=\begin{pmatrix}10 & 1\\ 2a-4 & -a-1\end{pmatrix}\begin{pmatrix}5 & 0\\ -2 & 2\end{pmatrix}$ $=\begin{pmatrix}48 & 2\\ 12a-18 & -2a-2\end{pmatrix}$ $\mathbf{BC}=\begin{pmatrix}2 & -1\\ 4 & 1\end{pmatrix}\begin{pmatrix}5 & 0\\ -2 & 2\end{pmatrix}=\begin{pmatrix}12 & -2\\ 18 & 2\end{pmatrix}$ $\mathbf{A}(\mathbf{BC})=\begin{pmatrix}1 & 2\\ a & -1\end{pmatrix}\begin{pmatrix}12 & -2\\ 18 & 2\end{pmatrix}$ $=\begin{pmatrix}48 & 2\\ 12a-18 & -2a-2\end{pmatrix}=(\mathbf{AB})\mathbf{C}$ (which demonstrates associativity of matrix multiplication)	M1 (AO 3.1a) A1 (AO 2.1) M1 (AO 1.1) A1 (AO 2.1) [4]	Finding **AB** (or **BC**) Finding (**AB**)**C** (or **A**(**BC**)) Finding **BC** (or **AB**) Correct final matrix and statement of equality		
		b	$\mathbf{AC}=\begin{pmatrix}1 & 2\\ a & -1\end{pmatrix}\begin{pmatrix}5 & 0\\ -2 & 2\end{pmatrix}=\begin{pmatrix}1 & 4\\ 5a+2 & -2\end{pmatrix}$ $\mathbf{CA}=\begin{pmatrix}5 & 0\\ -2 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ a & -1\end{pmatrix}=\begin{pmatrix}5 & 10\\ 2a-2 & -6\end{pmatrix}\neq\mathbf{AC}$ (so matrix multiplication is not commutative)	M1 (AO 1.1) A1 (AO 2.1) [2]	Finding **AC** (or **CA**) Finding the other and statement of non-equality		
		c	$\begin{pmatrix}1 & 2\\ a & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x+2y\\ ax-y\end{pmatrix}$ $x + 2y = 3x \Rightarrow y = x$ $ax - y = 3y$ and $y = x \Rightarrow a = 4$	M1 (AO 3.1a) A1 (AO 2.2a) A1 (AO 2.2a) [3]	Multiplying the vector into the matrix using the correct procedure		
			Total	9			

Question			Answer/Indicative content	Marks	Part marks and guidance	
7			Base case: For n = 1, $7^n + 3^{n-1}$ =7+1 = 8 which is a mulitple of 4. Assume that $f(k) = 7^k + 3^{k-1} = 4\lambda$ for some integer, λ $f(k+1) = 7^{k+1} + 3^k = 7.7^k + (7 - 4)3^{k-1}$ $= 7f(k) - 4.3^{k-1} = 7.4\lambda - 4.3^{k-1}$ $= 4(7\lambda - 3^{k-1}) = 4\lambda'$ Where λ ' is an integer because λ is and so rhs is a multiple of 4. So if true for n = k, true also for n = k + 1. But it is true for n = 1 Therefore true for all positive integers	B1 (AO 2.1) M1 (AO 2.1) M1 (AO 2.1) A1 (AO 2.5) A1 (AO 2.2a) [5]	Alternatively: $f(k+1) = 7^{k+1} + 3^k = 7.7^k + 3.3^{k-1}$ $= 7(4k - 3^{k-1}) + 3.3^{k-1} = \ldots$	
			Total	5		

Question			Answer/Indicative content	Marks	Part marks and guidance		
8			$\mathbf{M}^1 = \begin{pmatrix} 1 & 3(2^{1+1}-2) \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 3\times 2 \\ 0 & 2 \end{pmatrix}$ $= \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$ = **M**	B1(AO3.1a)	Full details must be shown.	DR: Detailed reasoning required for this question	
			Assume true for $n = k$ ie $\mathbf{M}^k = \begin{pmatrix} 1 & 3(2^{k+1}-2) \\ 0 & 2^k \end{pmatrix}$.	M1(AO2.1)	Must have statement in terms of some other variable than n.	Watch out for 2^{k+1} appearing prematurely as bottom right element	
			$\mathbf{M}^{k+1} = \mathbf{M}\mathbf{M}^k = \begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3(2^{k+1}-2) \\ 0 & 2^k \end{pmatrix}$	M1(AO1.1)	Uses inductive hypothesis properly	or $\mathbf{M}^k\mathbf{M} = \begin{pmatrix} 1 & 3(2^{k+1}-2) \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$	
			$\begin{pmatrix} 1 & 3(2^{k+1}-2)+6\times 2^k \\ 0 & 2\times 2^k \end{pmatrix}$	M1(AO1.1)	Genuine attempt at matrix multiplication (ie columns multiplied into rows)	$\begin{pmatrix} 1 & 6+2\times 3(2^{k+1}-2) \\ 0 & 2^k\times 2 \end{pmatrix}$	
			$\begin{pmatrix} 1 & 3(2^{k+1}-2)+3\times 2^{k+1} \\ 0 & 2\times 2^k \end{pmatrix} = \begin{pmatrix} 1 & 3(2^{(k+1)+1}-2) \\ 0 & 2^{(k+1)} \end{pmatrix}$	A1(AG)(AO2.2a)	Simplification with sufficient working to establish truth for $k + 1$	Must see at least one stage of working between expression for $\mathbf{M}^k\mathbf{M}$ (or $\mathbf{M}\mathbf{M}^k$) and required expression for $\mathbf{M}^{k+1}$ (e.g.: $\begin{pmatrix} 1 & 3(2+2\times 2^{k+1}-4) \\ 0 & 2^k\times 2 \end{pmatrix} = \begin{pmatrix} 1 & 3(2^{(k+2)}-2) \\ 0 & 2^{(k+1)} \end{pmatrix}$	
			So true for $n = k \Rightarrow$ true for $n = k + 1$. But true for $n = 1$. So true for all positive integer n	E1(AO2.4)	Clear	A formal proof by induction is required for full marks.	
				[6]			

Question			Answer/Indicative content	Marks	Part marks and guidance		
					conclusion for induction process. Needs a **fully correct proof** which usually means all other marks awarded. Cannot be awarded if there are mistakes in the proof.	**SC** If **B1M1M1M1** gained, but **A0** because of lack of intermediate step of working then allow **SC B1** for fully correct ending statement	
					<u>**Examiner's Comments**</u> This question was generally answered well, and candidates appeared to be confident with the idea of proof by induction. The most common mistakes were: * Not finishing off the proof by induction correctly, usually by stating 'since it is true for n = 1, n = k and n = k+1 …' * Failing to show sufficient detail to justify the n = 1 case * Failing to show sufficient detail to show that the n = k+1 case is true given that the n = k case is true * Confusion when multiplying 2×2^k which often was put equal to 4^k		
			Total	6			

Question			Answer/Indicative content	Marks	Part marks and guidance	
9			Formula is true for $n = 1$ since $\frac{1}{5^1} = \frac{1}{5}$ and $\frac{5 - 4 \times 1}{5} = \frac{1}{5}$	B1(AO 2.5)	Must include an arithmetic justification	
			Assume that the formula is true for $n = k$ Then $\sum_{r=1}^{k+1} \frac{5-4r}{5} = \frac{k}{5^k} + \frac{5-4(k+1)}{5^{k+1}}$	M1(AO 3.1a)	Add extra term	
			$= \frac{5k+5-4(k+1)}{5^{k+1}}$	M1(AO 2.1)	Manipulate	
			$= \frac{k+1}{5^{k+1}}$	A1(AO 2.2a)		
			So if true for $n = k$ then true also for $n = k + 1$ So true generally	E1(AO 2.4)	Needs conclusion	
				[5]		
			Total	5		

Question			Answer/Indicative content	Marks	Part marks and guidance		
10			$n = 4$: $4! = 24 \geq 6\times4 = 24$	B1(AO2.1)	Clear demonstration of equality is sufficient		
			Assume true for $n = k$ ie $k! \geq 6k$	M1(AO2.1)	Must have statement in terms of some other variable than n		
			$(k + 1)! = (k + 1)k! \geq (k + 1) \times 6k$	M1(AO1.1)	Uses inductive hypothesis properly		
			$\geq 6(k + 1)$ since $k \geq 1$	A1(AO2.2a)	Sufficient working to establish truth for $k + 1$	Must justify with $k \geq 1$ or $k \geq 4$ oe	
			So true for $n = k \Rightarrow$ true for $n = k + 1$. But true for $n = 4$. So true for all $n \geq 4$	E1(AO2.4) [5]	Clear conclusion for induction process.	A formal proof by induction is required for full marks.	
			Total	5			

Question			Answer/Indicative content	Marks	Part marks and guidance	
11				B1	Show clearly that result is true when $n = 1$	
			$\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}$	M1*	Add correct ($n + 1$)th term to given result	
			$\frac{n(2n+3)+1}{(2n+1)(2n+3)}$	DM1	Express as a single fraction with a correct denominator	
			$\frac{n+1}{2n+3}$	A1	Show correct factorisation and obtain correct simplified answer	
				B1	Clear statement of induction conclusion, previous 4 marks must be earned. Must include somewhere "true for $n = 1$", "true for n implies true for $n + 1$", "true for all n"	
				[5]	**Examiner's Comments** Some candidates failed to show clearly that the result is true for $n = 1$. Most added the correct next term to the given sum, although a significant minority added $\frac{1}{3}$ 'the value of the first term to the given sum. A significant number of candidates did not give a clear explanation of the induction process, so the last mark was often lost.	
			Total	5		

Question			Answer/Indicative content	Marks	Part marks and guidance		
12		a	DR $n = 1$, LHS = RHS = $\begin{pmatrix}1\\1\\1\end{pmatrix}$	B1 (AO 2.1)	Not just LHS = RHS.		
			Assume true for $n = k$ ie $\mathbf{a}_k = \left(-\frac{7}{8}\right)^{k-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$.	M1 (AO 2.1)	Must have statement in terms of some other variable than *n*.		
			$\mathbf{a}_{k+1} = (\mathbf{a}_k \times \mathbf{b}) \times \mathbf{b} = \left(\left(-\frac{7}{8}\right)^{k-1}\begin{pmatrix}1\\1\\1\end{pmatrix} \times \mathbf{b}\right) \times \mathbf{b}$	M1 (AO 2.1)	Uses inductive hypothesis properly		
			$\begin{pmatrix}1\\1\\1\end{pmatrix} \times \frac{1}{4}\begin{pmatrix}-3\\1\\2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1\\-5\\4\end{pmatrix}$	M1 (AO 1.1)			
			$\frac{1}{4}\begin{pmatrix}1\\-5\\4\end{pmatrix} \times \frac{1}{4}\begin{pmatrix}-3\\1\\2\end{pmatrix} = \frac{1}{16}\begin{pmatrix}-14\\-14\\-14\end{pmatrix} = -\frac{14}{16}\begin{pmatrix}1\\1\\1\end{pmatrix} = -\frac{7}{8}\begin{pmatrix}1\\1\\1\end{pmatrix}$	A1 (AO 1.1)	Any two components correct or all 3 wrong sign.		
			$\mathbf{a}_{k+1} = \left(-\frac{7}{8}\right)^{k-1} \times -\frac{7}{8}\begin{pmatrix}1\\1\\1\end{pmatrix}$	M1 (AO 2.1)	Working must be convincing. Do not award A1 if from 3wrong signs twice.		
			$\mathbf{a}_{k+1} = \left(-\frac{7}{8}\right)^{(k+1)-1}\begin{pmatrix}1\\1\\1\end{pmatrix}$	A1(AG) (AO 2.2a)	Simplification with sufficient working to establish truth for $k + 1$		
			So true for $n = k \Rightarrow$ true for $n = k + 1$. But true for $n = 1$. So true for all positive integer *n*	E1 (AO 2.4)		A formal proof by induction is required for full marks.	
				[8]			

Question			Answer/Indicative content	Marks	Part marks and guidance		
					Clear conclusion for induction process.		
		b	DR $\lvert \mathbf{a}_n \rvert = \left(\frac{7}{8}\right)^{n-1}\sqrt{3}$ or $0.875^{n-1}\sqrt{3}$ $n-1 > \dfrac{\ln\frac{0.001}{\sqrt{3}}}{\ln\left(\frac{7}{8}\right)}$ $(n=56.845...)$ So smallest value of n is 57	B1 (AO 3.1a) M1 (AO 1.1) A1 (AO 3.2a) [3]	Correctly taking logs (any base) and using 3rd law of logs with correct change of inequality sign		
			Total	11			
13			True for $k = 1$, since $1^5 - 1 = 0$ which is divisible by 5 Assume true for $n = k$, then $k^5 - k$ is divisble by 5 Then $(k + 1)^5 - (k + 1)$ $= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$ $= (5k^4 + 10k^3 + 10k^2 + 5k) + k^5 - k$ $= 5(k^4 + 2k^3 + 2k^2 + k) + k^5 - k$ So if divisible by 5 for $n = k$ then also for $n = k + 1$ True for $n = 1$, so true generally	B1 (AO2.1) M1 (AO2.1) A1 (AO2.5) A1 (AO2.2a) [4]			
			Total	4			

Question			Answer/Indicative content	Marks	Part marks and guidance		
14		a	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$	B1 (AO 3.1a)	$\Rightarrow b = 3$, $d = 4$		
			Determinant = $ad - bc = 5$	B1 (AO 3.1a)	$4a - 3c = 5$	0r det = –5 and follow through	
			$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$	B1 (AO 1.2)	Understanding of invariant point seen or implied		
			$(1 - a)x = by$ or $(1 - d)y = cx$	M1 (AO 2.2a)	May have $b = 3$ and/or $d = 4$ already substituted	$(1 - a)x = 3y$ or $-3y = cx$	
			$\dfrac{1-a}{c} = \dfrac{b}{1-d}$ Or $\frac{1-a}{b} = \frac{c}{1-d}$	M1 (AO 1.1)	Eliminating x and y		
			$c = a - 1$	A1 (AO 1.1)			
			e.g. $4a - 3(a - 1) = 5$	M1 (AO 1.1)	Attempting to solve their simultaneous equations	If no working and incorrect then M0A0.	
			$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$	A1 (AO 3.2a) [8]	Condone $a = 2$, etc as long as Matrix seen as $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$		
					<u>Examiner's Comments</u>		

Question			Answer/Indicative content	Marks	Part marks and guidance	
					This was found by the candidates to be the second hardest question part on the paper (after question 8(c)). Most candidates gained the first two marks, for using the image of (0,1) and for setting the determinant equal to 5 (those who set it equal to –5 or ±5 also gained this mark). Many candidates found the line of invariant points hard to interpret, many assuming that the matrix had an invariant line instead. Another common mistake was to make arithmetical mistakes when solving the simultaneous equations, such as sign errors. Some candidates stated the correct answer with very little working. As this question was not a “show that” and did not require “detailed reasoning” then this was allowed full credit. This is a risky strategy as if the answer was wrong then no marks for correct method can be given.	

Question			Answer/Indicative content	Marks	Part marks and guidance		
		b	Ne $\begin{pmatrix}3 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3x+y\\2x+2y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ ed $2x + y = 0$	M1 (AO 1.1) A1 (AO 2.2a) [2]	Substituting a general point into their matrix, calculating an image point and equating it to the object point. Final form must be $y = -2x$ or $x = -½y$ or a numerical multiple of $2x + y = 0$.	Need to have considered both x and y coordinates.	
					Examiner's Comments This part was found to be slightly easier than the other parts in question 8. The most common mistake was to try and find the invariant lines rather than the line of invariant points. A few candidates only considered the x coordinate and not the y coordinate. **Exemplar 4**		

Question			Answer/Indicative content	Marks	Part marks and guidance	
					$B = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} u \\ mu \end{pmatrix}$ $u' = 3u + mu$, $v' = 2u + 2mu$ $2u + 2mu = m(3u + mu)$ $2 + 2m = 3m + m^2$ $m^2 + m - 2 = 0$ $(m+2)(m-1) = 0$ $m = -2$ or $m = 1$ $y = x$ or $y = -2x$. Candidate has not used the definition of an invariant point (has use invariant lines of the form y=mx). If they went on to show that y=–2x is a line of invariant points and y=x is not this would gain credit. As it stands, no progress towards finding line of invariant points This candidate has confused "line of invariant points" with "an invariant line", and they have also assumed that all invariant lines have the form $y = mx$. A candidate using this approach can gain full credit if they then show that the line $y = -2x$ is an invariant line (i.e. by evaluating $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ -2a \end{pmatrix} = \begin{pmatrix} a \\ -2a \end{pmatrix}$)	
		c	$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ x+c \end{pmatrix} = \begin{pmatrix} 3x+x+c \\ 2x+2x+2c \end{pmatrix}$	M1* (AO 3.1a)		
			So $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 4x+c \\ 4x+2c \end{pmatrix}$ …	M1dep* (AO 2.2a)		
			…and $Y = X + c$	A1 (AO 1.1)	Could see the y component of the vector written as $4x + c + c$.	
			Alt $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix}$ $3x + mx + c = X$ $2x + 2mx + 2c = mX + c$	M1		

Question			Answer/Indicative content	Marks	Part marks and guidance
			$2x + 2mx + 2c = m(3x + mx + c) + c$ $x(m^2 + m - 2) + c(m - 1) = 0$ $x(m + 2)(m - 1) + c(m - 1) = 0$ If $m = 1, c(m - 1) = 0$ satisfied by any c	M1 A1 [3]	Need to eliminate X, i.e. an equation in x, m and c. __Examiner's Comments__ This was found to be the hardest question part in the paper, and it was also the one with the highest omit rate (20% did not attempt this question). Many candidates did not use the form of the line given in the question, but instead tried to use a general line such as $y = mx + c$ (or $y = mx$ but this was not a valid method). Candidates using $y = mx + c$ could gain full credit for this question, but considerably more work was needed than if they evaluated $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ x+c \end{pmatrix}$. Candidates got confused between lines of invariant points and invariant lines. In addition to this, many candidates assumed that an invariant line had to be of the form $y = mx$, i.e. that it passes through the origin.
			Total	13	

Question			Answer/Indicative content	Marks	Part marks and guidance	
15				B1	Show clearly result true for $n = 1$, accept $(u_1) = 2 \times 3 - 1 = 5$	
			$3(2 \times 3^n - 1) + 2$	M1	Substitute for u_n in recurrence relation	
				A1	Establish correct result for u_{n+1} convincingly	
				B1	Clear statement of induction conclusion, provided 1st 3 marks earned **Examiner's Comments** Some candidates failed to show clearly that the result is true for $n = 1$, while others started the process at $n = 2$. Most were able to show that using the given form in the recurrence relation set up the induction process correctly. A significant number of candidates did not give a clear explanation of the induction process, so the last mark was often lost.	
			Total	4		