Question	Scheme	Marks	AOs
1(a)	$\int \frac{1}{x^2+6x+25}\,dx = \int \frac{1}{(x+3)^2-9+25}\,dx = \int \frac{1}{(x+3)^2+16}\,dx$ or reaches integral in θ if using substitution.	M1	3.1a
	$= k\arctan\left(\frac{x+b}{a}\right)(+c)$ (or $k\theta$ where $4\tan\theta = x+3$)	M1	1.1b
	$= \frac{1}{4}\arctan\left(\frac{x+3}{4}\right)+c$	A1	1.1b
		(3)	
(b)	$\int_{-3}^{1}\left(1-\frac{25}{x^2+6x+25}\right)dx = \left[x-\frac{25}{4}\arctan\left(\frac{x+3}{4}\right)\right]_{-3}^{1} = (1-\ldots)-(-3-\ldots)$	M1	1.1b
	$= \left(1-\frac{25}{4}\arctan\left(\frac{4}{4}\right)\right)-\left(-3-\frac{25}{4}\arctan 0\right)$	A1ft	1.1b
	$= 4-\frac{25\pi}{16}$	A1	2.1
		(3)	
(c)	Since the graph crosses the x-axis at $x = 0$ the area lies partially below the x-axis,	**B1**	2.3
	hence the expression does not give the total area as the part below the axis counts as negative which cancels the positive area, so the student is not correct.	**B1**	
			(7 marks)

Notes:

(a)

M1: Identifies the need to and completes the square in the numerator to achieve a standard form, or selects the appropriate substitution $x + 3 = 4\tan\theta$. If using substitution, the integrand and dx must be dealt with and an integral in θ reached (or their chosen variable).

M1: Carries out the integration to a form $k\arctan\left(\frac{x+b}{a}\right)$

A1: Correct integral with or without c

(b)

M1: Applies limits to $x - 25\times$"their answer to (a)" and subtracts correct way.

A1ft: A correct unsimplified answer following through their answer to (a).

A1: Correct simplified exact answer.

(c)

B1: mentions that the graph crosses the x axis

B1: Mentions that part of the area is counted as negative with the correct conclusion.

Question	Scheme	Marks	AOs
2(a)	A correct method to sum the series, most likely by the method of differences. Look for $\frac{10}{r^2+8r+15}=\frac{A}{r+3}+\frac{B}{r+5}\Rightarrow A=..,\ B=..$ followed by an attempt at the sum (or with 1 instead of 10). (Induction may be attempted – see alt for (a).)	M1	3.1a
	$\frac{10}{r^2+8r+15}=\frac{5}{r+3}-\frac{5}{r+5}$ or $\frac{1}{r^2+8r+15}=\frac{1/2}{r+3}-\frac{1/2}{r+5}$	B1	1.1b
	$\sum_{r=1}^{n}\frac{10}{r^2+8r+15}=5\sum_{r=1}^{n}\left(\frac{1}{r+3}-\frac{1}{r+5}\right)$ $=5\left[\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{n+3}-\frac{1}{n+5}\right)\right]$	M1	2.1
	$=5\left(\frac{1}{4}+\frac{1}{5}-\frac{1}{n+4}-\frac{1}{n+5}\right)$	A1ft	1.1b
	$=5\left(\frac{5(n+4)(n+5)+4(n+4)(n+5)-20(n+5)-20(n+4)}{20(n+4)(n+5)}\right)=\ldots$	M1	2.1
	$=\frac{9n^2+41n}{4(n+4)(n+5)}$ (So $k=4$)	A1	1.1b
		(6)	
(b)	As $n\to\infty$, $T_n\to\frac{9}{4}$ or appropriate investigation tried.	M1	3.4
	Since the sum is increasing towards $\frac{9}{4}$ which is strictly less than 2.5 T_n can never reach 2.5, so the 2.5 million remaining tonnes of coal will not all be mined no matter how long the company keeps mining.	A1	3.2b
		(2)	
(c)	In the first 20 years $T_{20}=\frac{221}{120}$ million tonnes of coal have been mined, so $2.5-\frac{221}{120}=\frac{79}{120}$ tonnes remain.	M1	2.2b
	Hence $\frac{79}{120\times 20}$ extra tonnes per year need mining, so the new model is $M_r=\frac{79}{2400}+\frac{10}{r^2+8r+15}$.	A1ft	3.5c
		(2)	
		(10 marks)	

Notes:

(a)
M1: Attempts the sum using an appropriate method – ie method of differences. An attempt at partial fractions would evidence the attempt.
B1: Correct split into partial fractions.

M1: Applies method of differences showing evidence of the cancelling terms. The 5 may be missing at this stage and included later.

A1ft: Correct non-cancelling terms identified. Follow through their split into partial fractions if it leads to most terms cancelling.

M1: Puts the terms over a common denominator and simplifies. May be done in stages with the numerical fractions combined first etc, but look for appropriately adapted numerators for their method.
A1: Correct form with $k = 4$.

(b)

M1: Investigates the long term behaviour, e.g. by trying large values of n in the expression to see what happens, or by considering the long term limit.

A1: As scheme, comments that since the limit of the sum as $n \to \infty$ is 9/4 then the total amount of coal mined will never exceed 2.25 million tonnes, and so the coal will not all be mined even after a long time.

(c)

M1: Calculates the shortfall between 2.5 and the value of the sum at $n = 20$.

A1ft: Correct adaptation of the model adding (their shortfall)/20 to the original expression.

Alt (a)	Use of induction: Look for an attempt to find the value of k using $n = 1$ followed by an attempt at the inductive hypothesis.	M1	3.1a
	$n=1 \Rightarrow \dfrac{10}{1+8+15} = \dfrac{9+41}{k(5)(6)} \Rightarrow k = 4$	B1	1.1b
	Assume true for $n = p$, so $\sum_{r=1}^{p+1} M_r = \dfrac{9p^2+41p}{"4"(p+4)(p+5)} + \dfrac{10}{(p+1)^2+8(p+1)+15}$ $= \dfrac{9p^2+41p}{"4"(p+4)(p+5)} + \dfrac{10}{(p+4)(p+6)}$	M1	2.1
	$= \dfrac{(9p^2+41p)(p+6)+10\times"4"(p+5)}{"4"(p+4)(p+5)(p+6)}$	M1 A1ft	1.1b 1.1b
	$= \dfrac{9p^3+95p^2+286p+200}{4(p+4)(p+5)(p+6)} = \dfrac{(p+4)(p+1)(9p+50)}{4(p+4)(p+5)(p+6)}$ $= \dfrac{(p+1)\left[9(p+1)^2+41\right]}{4((p+1)+4)((p+1)+5)}$ Hence true for $n = 1$ (with $k = 4$) and if true for $n = p$ then true for $n = p+1$ so true for all positive integers n.	A1	2.1
		(6)	

M1: For use of induction look for an attempt to find the value of k first, followed by an attempt at proving the inductive step.
B1: Deduces $k = 4$.

M1: Assumes true for some p and uses their k in the expression for T_p (may use k instead of p, which is fine if there is no confusion as they have a value in the expression).

M1: Attempts to combine over a common denominator.

A1ft: Correct single fraction expression, follow through their k.
A1: Completes the induction step and make a suitable conclusion.

3.

Question	Scheme	Marks	AOs
3(a)	$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & k & 2 \end{vmatrix} = 3(3\times 2 - k\times -1) - 2(2\times 2 - 1\times -1) + 1(2\times k - 1\times 3)$	M1	1.1b
	$= 5k + 5$	A1	1.1b
		(2)	
(b)	(i) $3x + 2y + z = 4$	B1	1.1b
	(ii) EITHER $y = 2 - \lambda \Rightarrow \lambda = 2 - y$ OR $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = 0$ and $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0 = 0 \Rightarrow \mathbf{n} = A\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$	M1	1.1b
	EITHER $x = 1 + (2 - y) + \mu \Rightarrow z = 3 - (2 - y) + 2(x - (2 - y) - 1)$ OR $d = A\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5A$	M1	1.1b
	$\Rightarrow 2x + 3y - z = 5$	A1	1.1b
		(4)	

(c)	The planes meet when all three equations are satisfied, so we can find where they meet by solving $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & k & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix}$	B1	3.1a
	If the planes form a sheaf, then they must share a common line. But if $k \neq -1$ the determinant of the matrix is non-zero, so the equation has unique solution and hence the planes would meet in a single point. Therefore, we must have $k = -1$.	B1	2.3
		(4)	
			(8 marks)

Notes:

(a)

M1: Attempts determinant. Correct $() - () + ()$ structure, but allow up to two slips in entries.

A1: Determinant is $5k + 5$.

(b)(i)

B1: Correct equation. (Accept multiples.)

(ii)

M1: Eliminates λ or μ from Cartesian equations OR attempts to find a vector normal to both direction vectors using scalar product (or cross product may be used).

M1: Eliminates the other variable from their equations OR uses the scalar product with $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and their normal to find *d*.

A1: Correct equation. (Accept multiples.)

(c)

B1: Makes the link between the Cartesian equations and the matrix in (a).

B1: Identifies a sheaf cannot be formed if the solution is unique and so the matrix must be singular to form a sheaf, and hence $k = -1$.

4.

Question	Scheme	Marks	AOs
7(i)	When $n = 1$, LHS = $\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$, RHS = $\begin{pmatrix} 3\times1+1 & -1 \\ 9\times1 & 1-3\times1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$. So the statement is true for $n = 1$	B1	2.2a
	Assume true for $n = k$, so $\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^k = \begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}$	M1	2.4
	Then $\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^{k+1} = \begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}\begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}$ or $\begin{pmatrix} 3k+1 & -k \\ 9k & 1-3k \end{pmatrix}\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}$	M1	2.1
	$= \begin{pmatrix} 4(3k+1)-9k & -4k-(1-3k) \\ 9(3k+1)-18k & -9k-2(1-3k) \end{pmatrix}$ or $\begin{pmatrix} 4(3k+1)-9k & -(3k+1)+2k \\ 36k+9(1-3k) & -9k-2(1-3k) \end{pmatrix}$	A1	1.1b
	$= \begin{pmatrix} 3(k+1)+1 & -(k+1) \\ 9(k+1) & 1-3(k+1) \end{pmatrix}$	A1	1.1b
	Hence the result is true for $n = k+1$. Since it is true for $n = 1$, and if true for $k = n$ then true for $n = k+1$, thus by mathematical induction the result holds for all $n \in$ □	A1 cso	2.4
		(6)	
(ii)	(a) $2^2 = 4 \not< 4 = 2^2$ OR $3^2 = 9 \not< 8 = 2^3$ OR $4^2 = 16 \not< 16 = 2^4$	B1	1.1b
	(b) The statement $2k+1 < k^2$ is not true for all positive integers.	B1	1.1b
	(c) The statement in line 4 is true for positive integers $k > 2$ so the induction hypothesis is true for $n > 2$. So the induction holds from any base case greater than 2.	M1	2.3
	Since the result is true for $n = 5$ as $5^2 = 25 < 32 = 2^5$ and $2k+1 < k^2$ also true for $k > 5$ so the induction holds with base case $n = 5$.	A1	2.4
	But not true for $n = 2$, 3 or 4 as $2^2 = 4 \not< 4 = 2^2$ and $3^2 = 9 \not< 8 = 2^3$ and $4^2 = 16 \not< 16 = 2^4$. Hence true for $n = 1$ and for $n \ldots 5$	A1	2.1
		(5)	
			(11 marks)

Notes:

(a)

B1: Shows the general form holds for $n = 1$.

M1: Makes the inductive assumption, assume true for $n = k$.

M1: Attempts the multiplication either way.

A1: Correct matrix in terms of k.

A1: Rearranged into correct form to show true for $k + 1$.

A1: Completes the inductive argument conveying **all** three underlined points or equivalent at some point in their argument.

5.

Question	Scheme	Marks	AOs
5(a)	$\frac{dy}{dx} = \frac{1}{\sinh^2 x + 1} \times ..$	M1	1.2
	$\frac{dy}{dx} = \frac{1}{\sinh^2 x + 1} \times \cosh x$	A1	1.1b
	$= \frac{\cosh x}{\cosh^2 x} = \text{sech}\, x$ or use of correct identity $\sinh^2 x + 1 = \cosh^2 x$ later in the proof.	B1	2.1
	E.g. $\frac{d^2y}{dx^2} = -\text{sech}\, x \tanh x$ or $\frac{d^2y}{dx^2} = -(\cosh x)^{-2} \times \sinh x$ or even $\frac{d^2y}{dx^2} = \frac{(\sinh x)(\sinh^2 x + 1) - (\cosh x)(2\sinh x \cosh x)}{\left(\sinh^2 x + 1\right)^2}$	M1	1.1b
	$\frac{d^3y}{dx^3} = -(-\text{sech}\, x \tanh x)(\tanh x) + (-\text{sech}\, x)(\text{sech}^2 x)$ (oe) or any valid attempt at the third derivative from their second derivative. E.g. $\frac{d^2y}{dx^2} = -\tanh x \frac{dy}{dx}$ then $\frac{d^3y}{dx^3} = -\text{sech}^2 x \frac{dy}{dx} - \tanh x \frac{d^2y}{dx^2}$	M1 A1	3.1a 1.1b
	E.g. $\frac{d^3y}{dx^3} = \text{sech}\, x \tanh^2 x - \text{sech}^3 x = \text{sech}\, x(1 - \text{sech}^2 x) - \text{sech}^3 x$ $= \text{sech}\, x - 2\text{sech}^3 x = \frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^3$ * **or** $\frac{d^3y}{dx^3} = -\text{sech}^2 x \frac{dy}{dx} - \tanh x \frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^3 + \tanh^2 x \frac{dy}{dx}$ $= (1 - \text{sech}^2 x)\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^3 = \frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^3$ *	A1*	2.1
		(7)	

(b)	$\frac{d^4y}{dx^4}=\frac{d^2y}{dx^2}-6\left(\frac{dy}{dx}\right)^2\times\frac{d^2y}{dx^2}$	M1 A1	1.1b 1.1b
	$\frac{d^5y}{dx^5}=\frac{d^3y}{dx^3}-12\left(\frac{dy}{dx}\right)\times\left(\frac{d^2y}{dx^2}\right)^2-6\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}$	M1 A1	2.1 1.1b
		(4)	
(c)	At $x=0, y=0, y'=1, y''=0, y^{(3)}=-1, y^{(4)}=0$ and $y^{(5)}=-1-1\times0^2-6\times1^2\times(-1)=5$	M1	1.1b
	So $y=y(0)+xy'(0)+\frac{x^2}{2!}y''(0)+\frac{x^3}{3!}y^{(3)}(0)+\frac{x^4}{4!}y^{(4)}(0)+\frac{x^5}{5!}y^{(5)}(0)+\ldots$ with their evaluated values.	M1	1.1b
	$y=x-\frac{x^3}{6}+\frac{x^5}{24}+\ldots$	A1	2.5
		(3)	
			(14 marks)

6.

Question	Scheme	Marks	AOs
8(a)	$-7.5 + 2(0) + \lambda(1.5) = 0 \Rightarrow \lambda = \cdots$	M1	3.3
	$\lambda = 5$ *	A1*	1.1b
		(2)	
(b)	Solves $m^2 + 2m + 5 = 0 \Rightarrow m = \ldots$	M1	3.1b
	$m = -1 \pm 2\mathrm{i}$	A1ft	1.1b
	$x = \mathrm{e}^{-t}(A\cos 2t + B\sin 2t)$	A1	1.1b
	Using the initial conditions, $x = 1.5, t = 0$ to find a constant $1.5 = \mathrm{e}^0(A\cos 0 + B\sin 0) \Rightarrow A = \ldots\{1.5\}$	M1	3.4
	$\frac{dx}{dt} = -\mathrm{e}^{-t}(A\cos 2t + B\sin 2t) + \mathrm{e}^{-t}(-2A\sin 2t + 2B\cos 2t)$	M1	1.1b
	Using the initial conditions, $v = 0, t = 0$ to find the second constant $0 = -\mathrm{e}^0(1.5\cos 0 + B\sin 0) + \mathrm{e}^0(-3\sin 0 + 2B\cos 0)$ $\Rightarrow B = \ldots\{0.75\}$	dM1	3.4
	$x = e^{-t}(1.5\cos 2t + 0.75\sin 2t)$	A1	1.1b
		(7)	
(c)	Substitutes $t = 4.5$ into their equation for x $(x = -0.0117\ldots)$	M1	3.4
	Compares their value of x with 0 and evaluates the model	A1ft	3.5a
		(2)	
(d)	e. g. Take into account air resistance	B1	3.5c
		(1)	
		(12 marks)	
Notes:			

(a) **M1:** Substitutes $\ddot{x} = -7.5$, $\dot{x} = 0$ and $x = 1.5$ into the differential equation to find a value for λ **A1*:** Correct solution only
(b) **M1:** Forms and solves the auxiliary equation **A1ft:** Correct solution to their auxiliary equation. Follow through on their value of λ only. **A1:** Correct complementary function **M1:** Uses the information from the model $x = 1.5, t = 0$ to find a constant **M1:** Differentiates to find an expression for the velocity **dM1:** Uses the information from the model, $v = 0, t = 0$ to find another equation for the constants. **A1:** Correct equation for displacement
(c) **M1:** Substitutes to find a value of x
A1ft: Any suitable comment consistent with their value e.g. a good model since only 1cm out

7.

Question Number	Scheme	Marks
7 **(a)**	$x = r\cos\theta = 3\sin 2\theta\cos\theta$	B1
	$\dfrac{dx}{d\theta} = 6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta = 0$	M1
	$2\cos\theta\left(\cos^2\theta - 2\sin^2\theta\right) = 0$	M1
ALT	For the 2 M marks:	
	$x = 6\sin\theta\cos^2\theta \Rightarrow \dfrac{dx}{d\theta} = 6\cos^3\theta - 12\sin^2\theta\cos\theta = 0$	
	$\tan\phi = \dfrac{1}{\sqrt{2}}$ *	A1* (4)
(b)	$\tan\phi = \dfrac{1}{\sqrt{2}} \Rightarrow \sin\phi = \dfrac{1}{\sqrt{3}},\ \cos\phi = \dfrac{\sqrt{2}}{\sqrt{3}}$	M1
	$R = 3\times 2\times\dfrac{1}{\sqrt{3}}\times\dfrac{\sqrt{2}}{\sqrt{3}} = 2\sqrt{2}$	A1 (2)
(c)	Area of sector $= \dfrac{1}{2}\int r^2 d\theta = \dfrac{9}{2}\int \sin^2 2\theta\, d\theta$	M1
	$= \dfrac{9}{2}\int_0^{\arctan\left(\frac{1}{\sqrt{2}}\right)} \dfrac{1}{2}(1-\cos 4\theta)\, d\theta$	M1
	$= \dfrac{9}{2}\left[\dfrac{1}{2}\left(\theta - \dfrac{1}{4}\sin 4\theta\right)\right]_0^{\arctan\frac{1}{\sqrt{2}}}$	M1A1
	$= \dfrac{9}{4}\left[\arctan\dfrac{1}{\sqrt{2}} - \dfrac{1}{4}\sin 4\left(\arctan\dfrac{1}{\sqrt{2}}\right) - 0\right]$	dM1

	$\sin 4\phi = 2\sin 2\phi\cos 2\phi = 4\sin\phi\cos\phi\left(2\cos^2\phi - 1\right)$ $= 4\times\frac{1}{\sqrt{3}}\times\frac{\sqrt{2}}{\sqrt{3}}\left(2\times\frac{2}{3}-1\right) = \frac{4\sqrt{2}}{9}$	M1
	Area of sector $= \frac{9}{4}\left(\arctan\frac{1}{\sqrt{2}} - \frac{1}{4}\times\frac{4\sqrt{2}}{9}\right) = \frac{9}{4}\arctan\frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{4}$	A1 (7)
		[13]

Question Number	Scheme	Marks
(a)		
B1	State $x = (r\cos\theta) = 3\sin 2\theta\cos 2\theta$ May be given by implication	
M1	Attempt to differentiate $x = r\cos\theta$ or $x = r\sin\theta$ Product rule must be used	
M1	Use a correct double angle formula **and** equate the derivative of $r\cos\theta$ to 0	
ALT	**M1** Attempt the differentiation of $x = r\cos\theta$ or $x = r\sin\theta$ using the product rule (after using a double angle formula) **M1** Use a correct double angle formula **and** equate the derivative of $r\cos\theta$ to 0	
A1*	Complete to the given answer and no extras with no errors in the working. Accept θ or ϕ All values seen must be exact	
(b)		
M1	Attempt exact values for $\sin\theta$ and $\cos\theta$ and use these to obtain a value for *R*. Values for $\sin\theta$ and/or $\cos\theta$ may have been seen in (a)	
A1	A correct, exact value for *R*, as shown or any equivalent. Award M1A1 for a correct exact answer	
(c)		
M1	Use of Area $= \frac{1}{2}\int r^2 \mathrm{d}\theta$ Limits not needed (ignore any shown)	
M1	Use the double angle formula to obtain $k\int \frac{1}{2}(1\pm\cos 4\theta)\mathrm{d}\theta$ Ignore any limits given This is NOT dependent NB: There are other, lengthy, methods of reaching this point	
M1	Attempt the integration $\cos 4\theta \to \pm\frac{1}{4}\sin 4\theta$ (Not dependent)	
A1	Correct integration of $1 - \cos 4\theta$	
dM1	Correct use of correct limits. Depends on second and third M marks 0 at lower limit need not be shown	
M1	Attempt an exact numerical value for $\sin 4\left(\arctan\frac{1}{\sqrt{2}}\right)$	
A1	Correct final answer. Award M1A1 for a correct exact final answer	