Probability Distributions
Discrete Random Variables

Starter:

Mr Phillips is investigating the variation in daily maximum gust, t kn, for Camborne in June to August 1987. He used the large data set to select a sample of size 23. Mr Phillips selected the data from a day at random between the 1st and the 4^{th} of June, then selected the data from every fourth day after that.

(a) State the sampling technique Mr Phillips used. Systematic random sampling.

(b) From your knowledge of the large data set, explain why this process may not generate a sample of size 23. not all the days are available.

(c) Write down the probability that:

(i) the 1^{st} of June **and** the 1^{st} of July were selected. 0

(ii) the 1^{st} of June **or** the 1^{st} of July were selected. $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

1, 2, 3, 4, 5 ..... 9, ... 13

$\Rightarrow 4n-3$

2, 6, 10, ...

$4n-2$

June 30 days

July 31 days

1^{st} July 31^{st} day.

$4n-3=31$

$4n=34$

$n=\frac{34}{4}$ (not a whole number)

0.25

$4n-3 \Rightarrow 1$

$4n-2$

$4n-1 \Rightarrow 31$

$4n$

Example 1a:

Describe the probability distribution of X using:

(i) a probability table

(ii) a probability mass function

in each of the following cases.

(a) X is the number of Heads obtained when tossing three fair coins.

i)

$X=x$	0	1	2	3
$p(X=x)$	0.125	0.375	0.375	0.125

$$(h+t)^3 = h^3 + 3h^2t + 3ht^2 + t^3$$

$\boxed{h=0.5}$ $\boxed{t=0.5}$

Sample space = {
hhh — 3 heads
hht
hth — 2 heads
thh
htt
tht — 1 head
tth
ttt — 0 head

ii)

$$p(X=x) = \begin{cases} 0.125 & x = 0, 3 \\ 0.375 & x = 1, 2 \end{cases}$$

$\Rightarrow$
$p(X=0) = 0.125$
$p(X=3) = 0.125$
$p(X=1) = 0.375$
$p(X=2) = 0.375$

Example 1b:

Describe the probability distribution of X using:

(i) a probability table

(ii) a probability mass function

in each of the following cases.

(b) X is the number of days in a month that is randomly selected from a year with 365 days.

$X=x$	28	30	31
$p(X=x)$	$1/12$	$4/12$	$7/12$

J	F	M	A	M	J	J	A	S	O	N	D
1	2	3	4	5	6	7	8	9	10	11	12
31	28	31	30	31	30	31	31	30	31	30	31

$$p(X=x) = \begin{cases} \frac{1}{12} & x=28 \\ \frac{4}{12} & x=30 \\ \frac{7}{12} & x=31 \end{cases}$$

Example 1c:

Describe the probability distribution of X using:

(i) a probability table

(ii) a probability mass function

in each of the following cases.

(c) X is the number of days in a month that is randomly selected from Camborne May-Oct 1987.

$X=x$	30	31	otherwise
$p(X=x)$	2/6	4/6	0

M	J	J	A	S	O
5	6	7	8	9	10
31	30	31	31	30	31

$$p(X=x)=\begin{cases} \frac{2}{6} & X=30 \\ \frac{4}{6} & X=31 \\ 0 & \text{otherwise} \end{cases}$$

Example 1d:

Describe the probability distribution of X using:

(i) a probability table

(ii) a probability mass function

in each of the following cases.

(d) X is the ~~larger~~ max of the two values obtained from rolling two fair six-sided dice.

$X=x$	1	2	3	4	5	6
$p(X=x)$	$1/36$	$3/36$	$5/36$	$7/36$	$9/36$	$11/36$

Dice 1

Dice 2

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

$$p(X=x) = \frac{2x-1}{36} \quad x = 1, 2, 3, 4, 5, 6$$

According to *Monopoly* rules, a player gets out of Jail by...

- Throwing doubles on any of his next three turns. If he succeeds in doing this he immediately moves forward the number of spaces shown by his doubles throw. Even though he has thrown doubles he does not take another turn.)
- Using the "Get Out of Jail Free" card if he has it
- Purchasing the "Get Out of Jail Free" card from another player and playing it
- Paying a fine of \$50 before he rolls the dice on either of his next two turns.

If the player does not throw doubles by his third turn he must pay the \$50 fine. He then gets out of Jail and immediately moves forward the number of spaces shown by his throw.

Question 1:

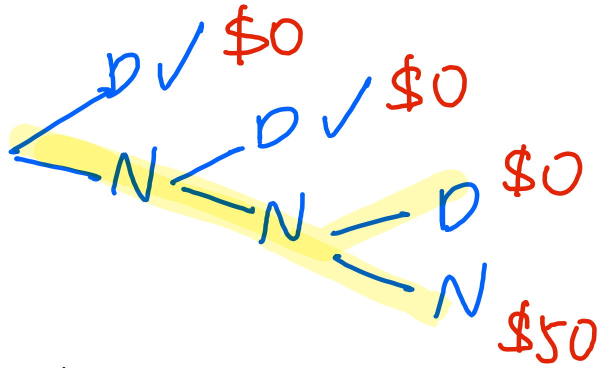
In a game of *Monopoly*, Mr Chan is in jail.
He throws two dice and do not want to pay the the \$50 fine.
He throws his two dice up to his third turn.

- The number of turns of Mr Chan takes to get out of jail is recorded as X.
- The fine that Mr Chan pays to get out of jail is recorded as Y.

(a) Show that $P(X = 1) = \cancel{\frac{1}{36}}$ $\frac{6}{36}$

(b) Find the probability distribution of X.

(c) Find the probability distribution of Y.



a) $\{1,1\}, \{2,2\}, \{3,3\}, \ldots, \{6,6\}$

$\Rightarrow p(\text{doubles}) = \frac{1}{6} \times \frac{1}{6} \times 6 = \frac{6}{36} = \frac{1}{6}$ //

$p(X=2) \Rightarrow$ not doubles in turn 1, doubles in turn 2

$\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$

$p(X=3) \Rightarrow \quad 1 - \frac{1}{6} - \frac{5}{36} = \frac{25}{36}$ (short way)

NND $= \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}$

NNN $= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$

$\Rightarrow \frac{25}{36}$ (Long way)

b)

X	1	2	3
$p(X=x)$	$1/6$	$5/36$	$25/36$

c)

Y	\$0	\$50
$p(Y=y)$	$91/216$	$125/216$

Question 2:

A biased tetrahedral die has faces numbered $0, 1, 2$ and 3. The die is rolled and the number face down on the die, X, is recorded. The probability distribution of X is

x	0	1	2	3
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$

- If $X = 3$ then the final score is 3
- If $X \neq 3$ then the die is rolled again and the final score is the sum of the two numbers.

The random variable T is the final score.

(a) Find $P(T = 2)$ $\frac{1}{12}$

(b) Find $P(T = 3)$ $\frac{23}{36}$

(c) Find the probability distribution of T.

(d) Given that the die is rolled twice, find the probability that the final score is 3. $\frac{5}{18}$

3 final score = 3

0,0	0
0,1	1
0,2	2
0,3	3
1,0	1
1,1	2
1,2	3
1,3	4
2,0	2
2,1	3
2,2	4
2,3	5

a) $P(T=2) = p(0,2) + p(1,1) + p(2,0)$

$= \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6}$

$= \frac{3}{36} \Rightarrow \frac{1}{12}$ //

b) $P(T=3) =$ $p(3)$

$p(0,3) + p(1,2) + p(2,1)$

$= \frac{1}{6} \times \frac{1}{2} + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6}$

$+ \frac{1}{2}$

$= \frac{23}{36}$ //

t	0	1	2	3	4	5
$P(T=t)$	$1/36$	$2/36$	$3/36$	$23/36$	$1/9$	$3/36$

Check $\Sigma p = 1$

$$\frac{1}{36}+\frac{2}{36}+\frac{3}{36}+\frac{23}{36}+\frac{1}{9}+\frac{3}{36}=1 \quad //$$

(a)

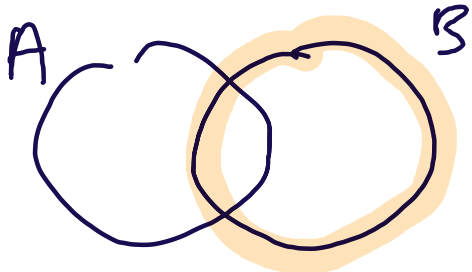
$$\frac{p(0,3)+p(1,2)+p(2,1)}{1/2} = \frac{23/36 - \frac{1}{2}}{\frac{1}{2}}$$

prob of rolled twice and T=3

prob of rolled twice

$$= \frac{5}{18} \quad //$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$



- Remember that $\sum P(X = x) = 1$ for any probability distribution.

Example 2:

Find the value of k for each of the following probability distributions:

(a) $P(X = x) = \begin{cases} \frac{k}{x} & x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$

(b) $P(X = x) = \begin{cases} kx^2 & x = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$

(c) $P(X = x) = \begin{cases} kx & x = 2, 4, 6 \\ k(x-2) & x = 8 \\ 0 & \text{otherwise} \end{cases}$

(d) $P(X = x) = \begin{cases} \frac{1}{4}k^x & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$

a)

$p(X=1) = \frac{k}{1}$

$p(X=2) = \frac{k}{2}$

$p(K=3) = \frac{k}{3}$

$p(K=4) = \frac{k}{4}$

$k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1$

$k = 12/25$

$X=x$	1	2	3	4
$p(X=x)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

(b)

$1k + 4k + 9k = 1$

$k = \frac{1}{14}$

(c)

$2k + 4k + 6k + 6k = 1$

$k = \frac{1}{18}$

(d)

$\frac{1}{4}k^0 + \frac{1}{4}k^1 + \frac{1}{4}k^2 = 1$

$k = \frac{-1+\sqrt{13}}{2}, \frac{-1-\sqrt{13}}{2}$

(only) (yes)

- **Discrete Uniform Distribution**
- X is said to be **uniformly distributed** if all of its possible outcomes have equal probability.

Example 3:

X is uniformly distributed across the values $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

The probability distribution of Y is given by

$$P(Y = y) = \begin{cases} \frac{1}{y} & y = 2, 3, k \\ 0 & \text{otherwise} \end{cases}$$

a) Given that X and Y are independent, find $P(X > Y)$.

This is a common type of exam question, using "sample space diagram"

c) $p(x > y \mid x = 8)$

d) $p(X > y \mid X = 3)$

a) $p(Y=y)$: $\frac{1}{2} + \frac{1}{3} + \frac{1}{k} = 1$

$k = 6$ //

$Y=y$	2	3	6
$p(Y=y)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$X=x$	1	2	3	4	5	6	7	8
$p(X=x)$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

$X=3 \quad y=2 \quad = \frac{1}{8}\times\frac{1}{2}$

$X=4 \quad y=2, y=3$

$X=5 \quad y=2, y=3$

$X=6 \quad y=2, y=3$

$X=7 \quad y=\text{anything}$

$X=8 \quad y=\text{anything}$

$\left(\frac{1}{8}\times\frac{1}{2}+\frac{1}{8}\times\frac{1}{3}\right)\times 3$

$+$

$3\times\frac{1}{8}\times\frac{5}{6}$

$+$

$2\times\frac{1}{8}\times 1$

$= 0.625$

$= \frac{5}{8}$ //

(b)(i) $X=\{1,2,3,4,5,6,7,8\}$

$Y=\{2,3,6\}$

$P(Y=2)=\frac{1}{2}$ $\quad \boxed{Y=2} \Rightarrow \boxed{X=3,4,5,6,7,8} \rightarrow \frac{1}{2}\times\frac{6}{8}$ $\quad 6/8$

$P(Y=3)=1/3$ $\quad \boxed{Y=3} \Rightarrow \boxed{X=4,5,6,7,8} \rightarrow \frac{1}{3}\times\frac{5}{8}$ $\quad 5/8$

$P(Y=6)=1/6$ $\quad \boxed{Y=6} \Rightarrow \boxed{X=7,8} \rightarrow \frac{1}{6}\times\frac{2}{8}$ $\quad 2/8$

$\therefore P(X>Y) = \frac{1}{2}\times\frac{6}{8}+\frac{1}{3}\times\frac{5}{8}+\frac{1}{6}\times\frac{2}{8} \boxed{=\frac{5}{8}}$

(c) $p(x>y \mid x=8)$

$$= \frac{\frac{1}{8}}{1/8} = 1$$

(d) $p(X>y \mid x=3)$

$$= \frac{1/8\times\frac{1}{2}}{1/8} = \frac{1/16}{1/8} = 1/2$$ //

Example 4:

The discrete random variable D has the following probability distribution where k is a constant.

d	10	20	30	40	50
$P(D = d)$	$\frac{k}{10}$	$\frac{k}{20}$	$\frac{k}{30}$	$\frac{k}{40}$	$\frac{k}{50}$

(a) Find the value of k as an exact fraction.

The random variables D_1 and D_2 are independent and each have the same distribution as D.

(b) Find $P(D_1 + D_2 = 80)$, giving your answer to 3 significant figures.

a)

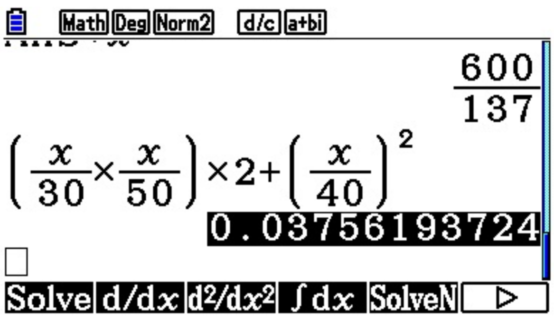
$$\frac{k}{10} + \frac{k}{20} + \frac{k}{30} + \frac{k}{40} + \frac{k}{50} = 1 \Rightarrow \boxed{k = \frac{600}{137}}$$

$$k\left(\frac{1}{10} + \frac{1}{20} + \frac{1}{30} + \frac{1}{40} + \frac{1}{50}\right) = 1$$

b)

$$p(30,50) + p(50,30) + p(40,40)$$

$$= 0.0376$$



Example 6:

A biased spinner can only land on one of the numbers 1, 2, 3 or 4. The random variable X represents the number that the spinner lands on after a single spin, and $P(X = r) = P(X = r + 2)$ for $r = 1, 2$.

It is given that $P(X = 2) = 0.35$.

(a) Find the complete probability distribution of X.

(b) Find the probability of obtaining a total score of 5 when the spinner is spun three times.

The random variable $Y = \frac{12}{X}$.

(c) Find $P(Y - X \leq 4)$.

$p(X=1) = p(X=3)$

$p(X=2) = p(X=4) = 0.35$

a)

$X=x$	1	2	3	4
$p(X=x)$	p	0.35	p	0.35
	0.15	0.35	0.15	0.35

$\Rightarrow p + 0.35 + p + 0.35 = 1$

$p = 0.15$

---

b)

total = 5

$\left.\begin{matrix} 1,1,3 \\ 1,3,1 \\ 3,1,1 \end{matrix}\right\} = 3 \times (0.15)^2 \times (0.15)$

$+$

$\left.\begin{matrix} 2,2,1 \\ 2,1,2 \\ 1,2,1 \end{matrix}\right\} = 3 \times (0.35)^2 \times (0.15)$

$\rightarrow 0.06525$

c)

Way1 Long way

$Y=$	$\frac{12}{1}$	$\frac{12}{2}$	$\frac{12}{3}$	$\frac{12}{4}$
$Y=$	12	6	4	3
prob	0.15	0.35	0.15	0.35

$X=x$	1	2	3	4
$p(X=x)$	p	0.35	p	0.35
	0.15	0.35	0.15	0.35

$p(Y-X \leq 4)$

Y	X	$Y-X$	
12	1	11	X
6	2	4	✓
4	3	1	✓
3	4	−1	✓

$0.35+0.15+0.35$
$=0.85$

$p(Y-X\leq 4) = 0.85$

The random variable $Y = \frac{12}{X}$.

(c) Find $P(Y - X \leq 4)$.

Way 2 short way

"Just sub it in"

$$P\left(\frac{12}{X} - X \leq 4\right)$$

consider

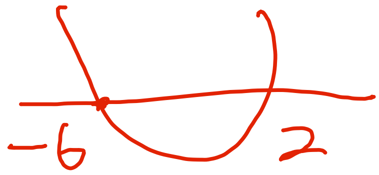
$\frac{12}{X} - X \leq 4$ (in this case $X = 1, 2, 3, 4$)

so multiplying by X will not change "sign"

$12 - X^2 \leq 4X$

$x^2 + 4x - 12 \geq 0$

$(X+6)(X-2) \geq 0$



$X \leq -6 \quad X \geq 2$

$P(X \leq -6)$ OR $P(X \geq 2)$

$\downarrow \qquad\qquad \downarrow$

$0 \qquad\qquad 0.85\%$

Example 7:

In a game, a player can score 0, 1, 2, 3 or 4 points each time the game is played. The random variable S, representing the player's score, has the following probability distribution where a, b and c are constants.

s	0	1	2	3	4
$P(S = s)$	a	b	c	0.1	0.15

The probability of scoring less than 2 points is twice the probability of scoring at least 2 points. Each game played is independent of previous games played.

Ms Esteban plays the game twice and adds the two scores together to get a total. Calculate the probability that the total is 6 points.

$$2(c+0.1+0.15) = a+b \quad -(1)$$

$$a+b+c+0.1+0.15=1 \quad -(2)$$

$$a+b = 1-0.25-c$$

$$a+b = 0.75-c \quad \Rightarrow \text{sub into } (1)$$

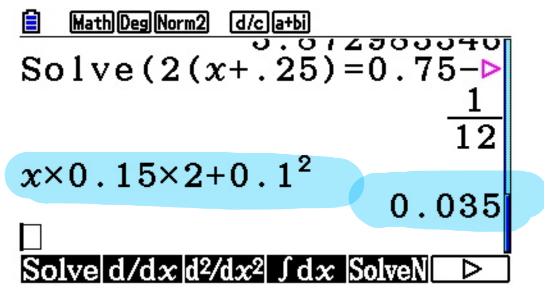
$$2(c+0.25) = 0.75-c$$

$$c = 1/12$$

$$p(2,4) = c(0.15)$$

$$p(3,3) = 0.1(0.1)$$

$$p(4,2) = 0.15(c)$$



Example 8:

Coin A is biased with $P(\text{heads}) = 0.6$. Coins B,C,D, and E are fair.

Each of the 5 coins are tossed once, and the number of heads obtained is X.

(a) Show that $P(X = 0) = 0.025$.

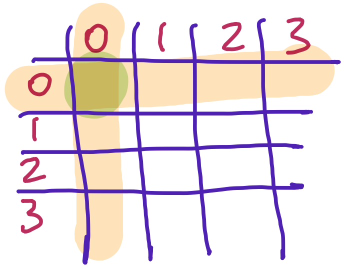
(b) Show that $P(X = 1) = 0.1375$.

The table below shows the probability distribution of X.

	0	**1**	**2**	**3**	**4**	**5**
P($X = x$)	0.025	0.1375	0.3	0.325	0.175	0.0375

Each of the 5 coins are tossed three times.

(c) Find the probability that the total number of heads obtained is 3.



Adapted Exam Questions:

Question 1

Tetrahedral dice have four faces. Two fair tetrahedral dice, one red and one blue, have faces numbered $0, 1, 2$, and 3 respectively. The dice are rolled and the numbers face down on the two dice are recorded. The random variable R is the score on the red die and the random variable B is the score on the blue die.

(a) Find $P(\{R = 3\} \cap \{B = 0\})$ $\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ **[2]**

(b) Find $P(\{R = 3\} \cup \{B = 0\})$ $\frac{1}{4} + \frac{1}{4} - \left(\frac{1}{4} \times \frac{1}{4}\right) = \frac{7}{16}$ $P(A) + P(B) - P(A \cap B) = P(A \cup B)$ **[2]**

The random variable T is R multiplied by B.

(c) Complete the diagram below to represent the sample space that shows all the possible values of T.

3	0	3	6	9
2	0	2	4	6
1	0	1	2	3
0	0	0	0	0
B / *R*	**0**	**1**	**2**	**3**

Sample space diagram of *T*

[3]

(d) The table below represents the probability distribution of the random variable T.

t	0	1	2	3	4	6	9
P(T=t)	a	b	1/8	1/8	c	1/8	d

Find the values of a, b, c and d. **[3]**

$a = \frac{7}{16}$ $b = \frac{1}{16}$ $c = \frac{1}{16}$ $d = \frac{1}{16}$

Question 2

The discrete random variable X has the probability distribution

x	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$4k$

(a) Show that $k = 0.1$

[1]

Two independent observations X_1 and X_2 are made of X.

(b) Show that $P(X_1 + X_2 = 4) = 0.1$

[2]

(c) Complete the probability distribution table for $X_1 + X_2$

y	2	3	4	5	6	7	8
$P(X_1 + X_2 = y)$	0.01	0.04	0.10		0.25	0.24	

[2]

(d) Find $P(1.5 < X_1 + X_2 \leq 3.5)$

[2]

Homework:

- Ex.6A: Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13