STEP 2 Mathematics 2019 Question 11 Combinatorics, Probability & Series

STEP 2 Mathematics 2019 Question 11 Combinatorics, Probability & Series

Section C: Probability and Statistics

11 **(i)** The three integers n_1, n_2 and n_3 satisfy $0 < n_1 < n_2 < n_3$ and $n_1 + n_2 > n_3$. Find the number of ways of choosing the pair of numbers n_1 and n_2 in the cases $n_3 = 9$ and $n_3 = 10$.

Given that $n_3 = 2n+1$, where n is a positive integer, write down an expression (which you need not prove is correct) for the number of ways of choosing the pair of numbers n_1 and n_2. Simplify your expression.

Write down and simplify the corresponding expression when $n_3 = 2n$, where n is a positive integer.

(ii) You have N rods, of lengths $1, 2, 3, \ldots, N$ (one rod of each length). You take the rod of length N, and choose two more rods at random from the remainder, each choice of two being equally likely. Show that, in the case $N = 2n+1$ where n is a positive integer, the probability that these three rods can form a triangle (of non-zero area) is

$$\frac{n-1}{2n-1}.$$

Find the corresponding probability in the case $N = 2n$, where n is a positive integer.

(iii) You have $2M+1$ rods, of lengths $1, 2, 3, \ldots, 2M+1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non-zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

Note: $\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1)$.

11 **(i)** The three integers n_1, n_2 and n_3 satisfy $0 < n_1 < n_2 < n_3$ and $n_1 + n_2 > n_3$. Find the number of ways of choosing the pair of numbers n_1 and n_2 in the cases $n_3 = 9$ and $n_3 = 10$.

Given that $n_3 = 2n+1$, where n is a positive integer, write down an expression (which you need not prove is correct) for the number of ways of choosing the pair of numbers n_1 and n_2. Simplify your expression.

Write down and simplify the corresponding expression when $n_3 = 2n$, where n is a positive integer.

$n_1, n_2, n_3 \in \mathbb{N}$ $\quad 0 < n_1 < n_2 < n_3$ $\quad n_1 + n_2 > n_3$

---

1 x 9 0

2 8 9 1

3 8 9
3 7 9 2

4 8 9
4 7 9 3
4 6 9

5 8 9
5 7 9 3
5 6 9

6 8 9
6 7 9 2

7 8 9 1

8 x 9 0

$n_1 \quad n_2 \quad n_3$

$n_3 = 9$

$0+0+1+1+2+2+3+3$

$= \underline{12}$

$n_3 = 9 \mid 2(1+2+3) = 9$

1 x 10 0

2 9 10 1

3 9 10

3 8 10 2

4 9 10

4 8 10 3

4 7 10

5 9 10

5 8 10

5 7 10 4

5 6 10

6 9 10

6 8 10 3

6 7 10

7 9 10

7 8 10 2

8 9 10 1

9 x 10 0

$n_3 = \boxed{10}$

$0+1+1+2+2+3+3+4$

$2(1+2+3) + \boxed{4}$

$= \underline{\underline{16}}$

$n_3 = 2n+1$

$n=2$

$n_3 = 2n$

$n=3$

$n_3 = 2n+1$

$n=3$

5		6		7	
1x5	0	1x6	0	1x7	0
245	1	256	1	267	1
345	1	356	2	367	2
4x5	0	346		357	
		456	1	467	
		5x6	0	457	2
				567	1
				6x7	0

$n_3 = 2n+1$

$n = 4$

$2(1+2+3) = 12$

$n_3 = 2n$

$n = 5$

$2(1+2+3)+4$

8

1x8	0	578	2
278	1	568	
378	2	678	1
368		7x8	0
478			
468	3		
458			

$n_3 = 2n$

$n = 4$

$2(1+2)+3$

$n_3 = 2n+1$

$$(1+2+3+\cdots n-1)\times 2$$

$$= 2\sum_{1}^{n-1} n = \cancel{2}\,\frac{(a+l)n}{\cancel{2}}$$

$$= (1+n-1)(n-1)$$

$$= n(n-1)$$

$n_3 = 2n$

$$2(1+2+3+\cdots n-2)+n-1$$

$$= 2\sum_{1}^{n-2} n + (n-1)$$

$$= \frac{2}{2}(1+n-2)(n-2) + (n-1)$$

$$= (n-2)(n-1) + (n-1)$$

$n_3 = 2n$ $= (n-1)[n-2+1]$

$= (n-1)(n-1)$

$= (n-1)^2$

11 **(i)** The three integers n_1, n_2 and n_3 satisfy $0 < n_1 < n_2 < n_3$ and $n_1 + n_2 > n_3$. Find the number of ways of choosing the pair of numbers n_1 and n_2 in the cases $n_3 = 9$ and $n_3 = 10$.

Given that $n_3 = 2n+1$, where n is a positive integer, write down an expression (which you need not prove is correct) for the number of ways of choosing the pair of numbers n_1 and n_2. Simplify your expression.

Write down and simplify the corresponding expression when $n_3 = 2n$, where n is a positive integer.

$n_3 = 2n+1 \Rightarrow$ # of ways $= n(n-1)$

$n_3 = 2n \Rightarrow$ # of ways $= (n-1)^2$

$n_3 = 2n+1 \Rightarrow$ # of ways $= n(n-1)$

$n_3 = 2n \Rightarrow$ # of ways $= (n-1)^2$

(ii) You have N rods, of lengths $1, 2, 3, \ldots, N$ (one rod of each length). You take the rod of length N, and choose two more rods at random from the remainder, each choice of two being equally likely. Show that, in the case $N = 2n+1$ where n is a positive integer, the probability that these three rods can form a triangle (of non-zero area) is

$$\rightarrow \frac{n-1}{2n-1}. \quad \checkmark$$

Find the corresponding probability in the case $N = 2n$, where n is a positive integer.

1234

$\boxed{\begin{matrix} 24 & 5 \\ 34 & 5 \end{matrix}}$ ✓

12
13 14
23
24
34

C^4_2

123456

12 23 34 45
13 24 35
14 25
15

C^5_2

$C^{n-1}_2 = \frac{(n-1)(n-2)}{2!}$

odd:

$n_3 = N = 2n+1$

$n=2 \quad N=5$

$$\frac{\frac{n(n-1)}{(2n)(2n-1)}}{2!} = \frac{n(n-1)}{n(2n-1)}$$

$$= \frac{n-1}{2n-1} \,//$$

Similarly,

123456

even

$$\frac{(n-1)^2}{\frac{(2n-1)(2n-2)}{2}}$$

12	23	34	45
13	24	35	
14	25		
15			

6

$n_3 = N = 2n$

$n = 3 \quad N = 6$

$${}_5C_2$$

$$\frac{4}{10} = \frac{2}{5}$$

$$= \frac{(n-1)^2}{(2n-1)(n-1)} = \frac{n-1}{2n-1}$$

//

256
346
356
456

//

Section C: Probability and Statistics

11 **(i)** The three integers n_1, n_2 and n_3 satisfy $0 < n_1 < n_2 < n_3$ and $n_1 + n_2 > n_3$. Find the number of ways of choosing the pair of numbers n_1 and n_2 in the cases $n_3 = 9$ and $n_3 = 10$.

Given that $n_3 = 2n+1$, where n is a positive integer, write down an expression (which you need not prove is correct) for the number of ways of choosing the pair of numbers n_1 and n_2. Simplify your expression.

Write down and simplify the corresponding expression when $n_3 = 2n$, where n is a positive integer.

(ii) You have N rods, of lengths 1, 2, 3, ..., N (one rod of each length). You take the rod of length N, and choose two more rods at random from the remainder, each choice of two being equally likely. Show that, in the case $N = 2n+1$ where n is a positive integer, the probability that these three rods can form a triangle (of non-zero area) is

$$\boxed{\frac{n-1}{2n-1}}.$$

Find the corresponding probability in the case $N = 2n$, where n is a positive integer.

(iii) You have $2M+1$ rods, of lengths 1, 2, 3, ..., $2M+1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non-zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

Note: $\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1)$.

$\sum_{1}^{K} K = \frac{(K)(K+1)}{2}$

$n_3 = 2n+1 \Rightarrow$ # of ways $= n(n-1)$

$n_3 = 2n \Rightarrow$ # of ways $= (n-1)^2$

$C^{n-1}_{2} = \frac{(n-1)(n-2)}{2!}$ prob of (1 ..... N-1, given N is chosen:)

① odd $= \frac{n-1}{2n-1}$ ② even $= \frac{n-1}{2n-1}$

1 4 6 4 1

12345

1 5 10 10 5 1

123 234 345
124 235
125 245
134
135
145

$$\frac{C^4_2}{C^5_3} \times \frac{n-1}{2n-1}$$

123456

1 6 15 20 15 6 1

123 234 345 456
124 235 346
125 236 356
126 245
134 246
135 256
136
145
146
156

$$\frac{C^5_2}{C^6_3} \times \frac{n-1}{2n-1}$$

$$\left(C^6_2 \times \frac{n-1}{2n-1}\right)$$

$$\frac{1}{C^7_3}\left(C^5_2 \times \frac{n-1}{2n-1} + C^4_2 \times \frac{n-1}{2n-1} + C^3_2 \times \frac{n-1}{2n-1} + \cdots\right)$$

$M=3$

(iii) You have $2M+1$ rods, of lengths $1, 2, 3, \ldots, 2M+1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non-zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$$

Note: $\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1)$.

$$\sum_{n=1}^{M} \left(P(\text{odd}) \times \frac{n-1}{2n-1} + P(\text{even}) \times \frac{n-1}{2n-1} \right)$$

$$P(\text{odd}_{2n+1}) = \frac{{}^{2n}C_2}{{}^{2M+1}C_3} = \frac{2n(2n-1)/2!}{(2M+1)(2M)(2M-1)/3!}$$

$$P(\text{even}_{2n}) = \frac{{}^{2n-1}C_2}{{}^{2M+1}C_3} = \frac{(2n-1)(2n-2)/2!}{(2M+1)(2M)(2M-1)/3!}$$

$$\frac{3}{(2M+1)(M)(2M-1)} \sum_{n=1}^{n=M} \left(\frac{n-1}{\cancel{2n-1}} \left(n\cancel{(2n-1)} + \cancel{(2n-1)}(n-1) \right) \right)$$

$$\frac{3}{(2M+1)(M)(2M-1)} \sum_{1}^{M} \left((n-1)(2n-1) \right)$$

(iii) You have $2M+1$ rods, of lengths $1, 2, 3, \ldots, 2M+1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non-zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$$

Note: $\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1)$.

$$\sum_1^n n = \frac{n(n+1)}{2} \qquad \sum_1^n 1 = K$$

$$\frac{3}{(2M+1)(M)(2M-1)} \sum_1^M \left(2n^2 - 3n + 1 \right)$$

$$\frac{3}{(2M+1)(\cancel{M})(2M-1)} \left(\frac{2}{6}(\cancel{M})(M+1)(2M+1) - \frac{3}{2}(\cancel{M})(M+1) + \cancel{M}. \right)$$

$$= \frac{\cancel{3}\left(\frac{2M^2+3M+1}{3} - \frac{3\ M+1}{2} + 1 \right)}{(2M+1)(2M-1)} \frac{\times 2}{\times 2}$$

$$= \frac{2(2M^2+3M+1) - 3(3M+1) + 6}{2(2M+1)(2M-1)}$$

$$= \frac{4M^2+6M+2 - 9M - 3 + 6}{2(2M+1)(2M-1)}$$

(iii) You have $2M+1$ rods, of lengths $1, 2, 3, \ldots, 2M+1$ (one rod of each length), where M is a positive integer. You choose three at random, each choice of three being equally likely. Show that the probability that the rods can form a triangle (of non-zero area) is

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}.$$

$$\frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$$

Note: $\sum_{k=1}^{K} k^2 = \frac{1}{6}K(K+1)(2K+1)$.

$$\sum_{1}^{n} n = \frac{n(n+1)}{2} \qquad \sum_{1}^{n} 1 = K$$

$$= \frac{4M^2 - 3M + 5}{2(2M+1)(2M-1)}$$

$$= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)} \quad //$$