The equation $x^3 - 8x^2 + cx + d = 0$, where c and d are real numbers, has roots α, β, γ.

When plotted on an Argand diagram, the triangle with vertices at α, β, γ has an area of 8.

Given $\alpha = 2$, find the values of c and d.

Fully justify your solution.

(Total 5 marks)

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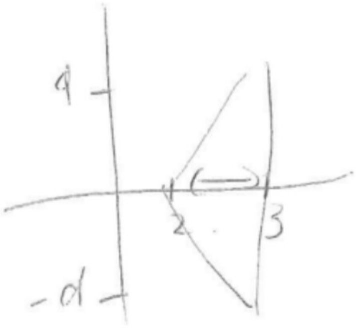
Fully justify your solution.

(Total 5 marks)

~~$2 \pm c \pm di$~~

~~[illegible]~~



$2 + a + bi + a - bi = 8$

$2a = 6$

$a = 3$

$\frac{2d \times 1}{2} = 8$

$d = 8$

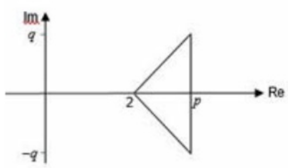
$d = -\alpha\beta\gamma$

$= -2(3+8i)(3-8i)$

$d = -146$

$c = 2(3+8i) + 2(3-8i) + (3+8i)(3-8i)$

c = 12 + 73 = 85

Marking Instructions	AO	Marks	Typical Solution
Writes β and γ in the form $p \pm qi$ (seen anywhere in the solution)	AO2.5	B1	Real coefficients $\Rightarrow \beta = p + qi$ and $\gamma = p - qi$
Uses "sum of the roots = −b/a" together with a conjugate pair to determine the real part (p) of β and γ	AO3.1a	M1	$\alpha + \beta + \gamma = 8$ $\Rightarrow 2 + p + qi + p - qi = 8$ $\Rightarrow 2 + 2p = 8$ $\Rightarrow p = 3$
Uses '(their p)' − 2 and the area of the triangle on an Argand diagram to determine the imaginary parts of β and γ	AO3.1a	M1	$(p-2)q = 8$ $\Rightarrow q = 8$ 
Uses a correct method to find the value of c or d using 'their' values of $p \pm qi$	AO1.1a	M1	$\beta = 3 + 8i$ and $\gamma = 3 - 8i$
Obtains correct values for c and d. CAO	AO1.1b	A1	$d = -\alpha\beta\gamma = -146$ $c = \sum\alpha\beta = 85$
			Total 5 marks

The cubic equation

$$9x^3 - 5x^2 + 4x + 7 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(3\alpha - 2)$, $(3\beta - 2)$ and $(3\gamma - 2)$, giving your answer in the form $aw^3 + bw^2 + cw + d = 0$, where a, b, c and d are integers to be determined.

(Total for question = 5 marks)

Question	Scheme	Marks	AOs
	$w=3x-2 \Rightarrow x=\frac{w+2}{3}$	B1	3.1a
	$9\left(\frac{w+2}{3}\right)^3-5\left(\frac{w+2}{3}\right)^2+4\left(\frac{w+2}{3}\right)+7=0$	M1	3.1a
	$\frac{1}{3}\left(w^3+6w^2+12w+8\right)-\frac{5}{9}\left(w^2+4w+4\right)+\frac{4}{3}(w+2)+7=0$		
	$3w^3+13w^2+28w+91=0$	dM1 A1 A1	1.1b 1.1b 1.1b
		(5)	
	Alternative:		
	$\alpha+\beta+\gamma=\frac{5}{9}, \alpha\beta+\beta\gamma+\alpha\gamma=\frac{4}{9}, \alpha\beta\gamma=-\frac{7}{9}$	B1	3.1a
	New sum $=3(\alpha+\beta+\gamma)-6=-\frac{13}{3}$	M1	3.1a
	New pair sum $=9(\alpha\beta+\beta\gamma+\gamma\alpha)-12(\alpha+\beta+\gamma)+12=\frac{28}{3}$		
	New product $=27\alpha\beta\gamma-18(\alpha\beta+\beta\gamma+\gamma\alpha)+12(\alpha+\beta+\gamma)-8=-\frac{91}{3}$		
	$w^3-\left(-\frac{13}{3}\right)w^2+\frac{28}{3}w-\left(-\frac{91}{3}\right)=0$	dM1	1.1b
	$3w^3+13w^2+28w+91=0$	A1 A1	1.1b 1.1b
		(5)	
			(5 marks)

Q2.

Given that $z = e^{i\theta}$

(a) show that $z^n + \frac{1}{z^n} = 2\cos n\theta$

where n is a positive integer.

(2)

(b) Show that

$$\cos^6\theta = \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right)$$

(5)

(c) Hence solve the equation

$$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta = 0 \qquad 0 \leqslant \theta \leqslant \pi$$

Give your answers to 3 significant figures.

(4)

Q2.

Given that $z = e^{i\theta}$

(a) show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

where n is a positive integer.

$z^n + z^{-n} = e^{in\theta} + e^{-in\theta} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$
$= 2\cos n\theta$

(b) Show that

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

$\left(z+\frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$
$= z^6 + z^{-6} + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$
$= z^6 + z^{-6} + 6(z^4+z^{-4}) + 15(z^2+z^{-2}) + 20$
$64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$
$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta ...$

(5)

(c) Hence solve the equation

$$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta = 0 \qquad 0 \leqslant \theta \leqslant \pi$$

Give your answers to 3 significant figures.

(4)

$32\cos^6\theta - 10 = 0$
$\cos^6\theta = \frac{5}{16}$
$\cos\theta = \pm\sqrt[6]{\frac{5}{16}}$
$\theta = 0.603,\ 2.54$

Question Number	Scheme	Marks
(a)	$z^n = e^{in\theta} = \cos n\theta + i\sin n\theta$	
	$\frac{1}{z^n} = e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta$	
	$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$ *	M1A1cso (2)
(b)	$\left(z+\frac{1}{z}\right)^6 = z^6 + 6z^5\times\frac{1}{z} + \frac{6\times5}{2!}z^4\times\frac{1}{z^2} + \frac{6\times5\times4}{3!}z^3\times\frac{1}{z^3} + \frac{6\times5\times4\times3}{4!}z^2\times\frac{1}{z^4} + \frac{6\times5\times4\times3\times2}{5!}z\times\frac{1}{z^5} + \frac{1}{z^6}$	M1A1
	$(2\cos\theta)^6 = z^6 + 6z^4 + 15z^2 + 20 + 15\times\frac{1}{z^2} + 6\times\frac{1}{z^4} + \frac{1}{z^6}$	
	$64\cos^6\theta = z^6 + \frac{1}{z^6} + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$	M1
	$64\cos^6\theta = 2\cos 6\theta + 6\times2\cos 4\theta + 15\times2\cos 2\theta + 20$	M1
	$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$ *	A1* (5)
(c)	$\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 = 10$	
	$32\cos^6\theta = 10$	M1A1
	$\cos\theta = \pm\sqrt[6]{\frac{5}{16}}$	
	$\theta = 0.6027..., 2.5388... \qquad \theta = 0.603,\ 2.54$	M1A1 (4)
(d)	$\int_0^{\frac{\pi}{3}}(32\cos^6\theta - 4\cos^2\theta)\,d\theta$	
	$= \int_0^{\frac{\pi}{3}}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 - 4\cos^2\theta)\,d\theta$	
	$= \int_0^{\frac{\pi}{3}}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10 - 2 - 2\cos 2\theta)\,d\theta$	M1
	$= \left[\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta + \frac{13}{2}\sin 2\theta + 8\theta\right]_0^{\frac{\pi}{3}}$	M1A1
	$= (0) + \frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right) + \frac{13}{2}\times\frac{\sqrt{3}}{2} + \frac{8\pi}{3} \quad (-0)$	dM1
	$= \frac{5\sqrt{3}}{2} + \frac{8\pi}{3}$ oe	A1 (5)
		[16]

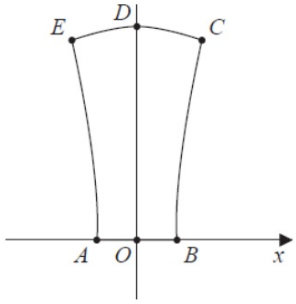


Figure 2

Figure 2 shows the vertical cross-section, *AOBCDE*, through the centre of a wax candle.

In a model, the candle is formed by rotating the region bounded by the y-axis, the line *OB*, the curve *BC*, and the curve *CD* through 360° about the y-axis.

The point *B* has coordinates (3, 0) and the point *C* has coordinates (5, 15).

The units are in centimetres.

The curve *BC* is represented by the equation

$$y = \frac{\sqrt{225x^2 - 2025}}{a} \qquad 3 \leqslant x < 5$$

where a is a constant.

(a) Determine the value of a according to this model.

(2)

The curve *CD* is represented by the equation

$$y = 16 - 0.04x^2 \qquad 0 \leqslant x < 5$$

(b) Using algebraic integration, determine, according to the model, the exact volume of wax that would be required to make the candle.

(9)

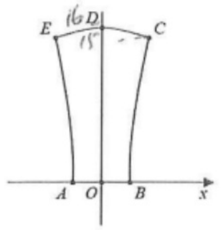
(c) State a limitation of the model.

(1)

When the candle was manufactured, 700 cm^3 of wax were required.

(d) Use this information and your answer to part (b) to evaluate the model, explaining your reasoning.

(1)



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...

a) $15 = \frac{\sqrt{225(5)^2 - 2025}}{a}$

$a = 4$

$V = \pi \int_0^{15} \frac{16y^2 + 2025}{225} \, dy + \pi \int_{15}^{16} \frac{16 - y}{0.04} \, d$

$= \frac{455\pi}{2}$

c) the surface of the candle will have bumps and so will not be perfectly smooth

d) $700 \approx 715$ so the

model is suitable as the estimate is close to the actual value

Question	Scheme	Marks	AOs
(a)	$(5, 15) \Rightarrow 15 = \frac{\sqrt{225 \times 5^2 - 2025}}{a} \Rightarrow a = ...$	M1	3.3
	$a = 4$	A1	1.1b
		(2)	
(b)	Evidence of the use of $\pi \int x^2 \, dy$ for the curve *BC* or the curve *CD*	M1	3.1b
	For *BC* $V_1 = \frac{\pi}{225} \int (16y^2 + 2025) \, dy$ or $\pi \int \left(\frac{16}{225} y^2 + 9 \right) dy$	A1ft	1.1b
	For *CD* $V_2 = 25\pi \int (16 - y) \, dy$ or $\pi \int (400 - 25y) \, dy$	A1	1.1b
	$V_1 = \frac{\pi}{225} \int_0^{15} (16y^2 + 2025) \, dy$ or $\pi \int_0^{15} \left(\frac{16}{225} y^2 + 9 \right) dy$	M1	3.3
	$V_2 = 25\pi \int_{15}^{16} (16 - y) \, dy$ or $\pi \int_{15}^{16} (400 - 25y) \, dy$	M1	3.3
	$V_1 = \frac{\{\pi\}}{225} \left[\frac{16y^3}{3} + 2025y \right]_0^{15}$ or $\{\pi\} \left[\frac{16y^3}{675} + 9y \right]_0^{15}$	A1ft	1.1b
	$V_2 = 25\{\pi\} \left[16y - \frac{y^2}{2} \right]_{15}^{16}$ or $\{\pi\} \left[400y - \frac{25y^2}{2} \right]_{15}^{16}$	A1ft	1.1b
	$V = V_1 + V_2 = \frac{\pi}{225}(18000 + 30375) + 25\pi \left(128 - \frac{255}{2} \right)$ $V = V_1 + V_2 = 215\pi + 12.5\pi$	M1	3.4
	$V = \frac{455\pi}{2}$ cm^3 or 227.5π cm^3	A1	2.2b
		(9)	
(c)	E.g. • The equation of the curve may not be a suitable model • The sides of the candle will not be perfectly curved/smooth • There will be a whole in the middle for the wick	B1	3.5b
		(1)	
(d)	Makes an appropriate comment that is consistent with their value for the volume and 700 cm^3. E.g. a good estimate as 700 cm^3 is only 15 cm^3 less than 715 cm^3	B1ft	3.5a
		(1)	
		(13 marks)	

$$\mathbf{A} = \begin{pmatrix} 2 & k & 2 \\ 2 & 2 & k \\ 1 & 2 & 2 \end{pmatrix} \qquad \text{where } k \text{ is a constant}$$

(a) Determine the values of k for which **A** is singular.

(2)

Given that **A** is non-singular,

(b) find $\mathbf{A}^{-1}$, giving your answer in terms of k.

(4)

(Total for question = 6 marks)

(4)

a) $2\begin{vmatrix}2 & k\\2 & 2\end{vmatrix} - k\begin{vmatrix}2 & k\\1 & 2\end{vmatrix} + 2\begin{vmatrix}2 & 2\\1 & 2\end{vmatrix}$

$2(4-2k) - k(4-k) + 2(4-2) = 0$

$k^2 - 8k + 12 = 0$

$(k-6)(k-2) = 0$

$k = 2,\ k = 6$

(Total for question = 6 marks)

$A^{-1} = \frac{1}{k^2-8k+12}\begin{pmatrix}4-2k & 4-2k & k^2-4\\ k-4 & 2 & 4-2k\\ 2 & k-4 & 4-2k\end{pmatrix}$

b) $\begin{pmatrix}\begin{vmatrix}2 & k\\2 & 2\end{vmatrix} & -\begin{vmatrix}2 & k\\1 & 2\end{vmatrix} & +\begin{vmatrix}2 & 2\\1 & 2\end{vmatrix}\\ -\begin{vmatrix}k & 2\\2 & 2\end{vmatrix} & +\begin{vmatrix}2 & 2\\1 & 2\end{vmatrix} & -\begin{vmatrix}2 & k\\1 & 2\end{vmatrix}\\ +\begin{vmatrix}k & 2\\2 & k\end{vmatrix} & -\begin{vmatrix}2 & 2\\2 & k\end{vmatrix} & +\begin{vmatrix}2 & k\\2 & 2\end{vmatrix}\end{pmatrix} = \begin{bmatrix}4-2k & k-4 & 2\\ 4-2k & 2 & k-4\\ k^2-4 & 4-2k & 4-2k\end{bmatrix}$

Question Number	Scheme	Notes	Marks
(a)	$\mathbf{A} = \begin{pmatrix}2 & k & 2\\2 & 2 & k\\1 & 2 & 2\end{pmatrix}$		
	$\lvert\mathbf{A}\rvert = 2(4-2k) - k(4-k) + 2(4-2) = 0$ $\Rightarrow k^2 - 8k + 12 = 0 \Rightarrow k = \ldots$ Attempts det $\mathbf{A} = 0$ and solves 3TQ to obtain 2 values for k Note that the usual rules for solving a 3TQ do not need to be applied as long as 2 values for k are obtained. The attempt at the determinant should be a correct expression for their row or column so allow errors only when collecting terms Note that the rule of Sarrus gives $8 + k^2 + 8 - 4 - 4k - 4k = 0$		M1
	$k = 2,\ 6$	Correct values.	A1
	Marks for part (a) can only be scored in their attempt at (a) and not recovered from part (b)		
			(2)
(b)	$\begin{pmatrix}2 & k & 2\\2 & 2 & k\\1 & 2 & 2\end{pmatrix} \rightarrow \begin{pmatrix}4-2k & 4-k & 2\\2k-4 & 2 & 4-k\\k^2-4 & 2k-4 & 4-2k\end{pmatrix} \rightarrow \begin{pmatrix}4-2k & k-4 & 2\\4-2k & 2 & k-4\\k^2-4 & 4-2k & 4-2k\end{pmatrix}$ Applies the correct method to reach at least a matrix of cofactors Should be an attempt at the minors followed by $\begin{pmatrix}+ & - & +\\- & + & -\\+ & - & +\end{pmatrix}$ If there is any doubt then look for at least 6 correct cofactors		M1
	$\begin{pmatrix}4-2k & k-4 & 2\\4-2k & 2 & k-4\\k^2-4 & 4-2k & 4-2k\end{pmatrix} \rightarrow \begin{pmatrix}4-2k & 4-2k & k^2-4\\k-4 & 2 & 4-2k\\2 & k-4 & 4-2k\end{pmatrix}$ dM1: Attempts adjoint matrix by transposing. Dependent on previous mark. A1: Correct adjoint		dM1 A1
	$\mathbf{A}^{-1} = \frac{1}{k^2-8k+12}\begin{pmatrix}4-2k & 4-2k & k^2-4\\k-4 & 2 & 4-2k\\2 & k-4 & 4-2k\end{pmatrix}$ Fully correct inverse or follow through their incorrect determinant **from part (a)** where their determinant is a function of k		A1ft
	Ignore any labelling of the matrices and allow any type of brackets around the matrices		
			(4)
			Total 6

Q5.

(a) Express $\frac{2}{r(r^2-1)}$ in partial fractions.

(3)

(b) Hence find, in terms of *n*,

$$\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$$

Give your answer in the form

$$\frac{n^2 + An + B}{Cn(n+1)}$$

where *A*, *B* and *C* are constants to be found.

(5)

(Total for question = 8 marks)

Q5.

(a) Express $\frac{2}{r(r^2-1)}$ in partial fractions.

$r(r+1)(r-1)$

(3)

(b) Hence find, in terms of n,

$$\sum_{r=2}^{n} \frac{1}{r(r^2-1)}$$

Give your answer in the form

$$\frac{n^2 + An + B}{Cn(n+1)}$$

where A, B and C are constants to be found.

(5)

(Total for question = 8 marks)

$\frac{A}{r} + \frac{B}{r-1} + \frac{C}{r+1}$

$-\frac{2}{r} + \frac{1}{r-1} + \frac{1}{r+1}$

$\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}$

$r=2 \quad \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$

$r=3 \quad \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$

$r=4 \quad \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$

$r=n-1 \quad \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}$

$r=n \quad \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$

b) $1 - 1 + \frac{1}{2}$

$\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}$

$\frac{n(n+1) - 2(n+1) + 2n}{2n(n+1)}$

$\frac{n^2+n-2n-2+2n}{2n(n+1)}$

$\frac{1}{2}\sum_{r=2}^{n} \frac{2}{r(r^2-1)} = \sum_{r=2}^{n} \frac{1}{r(r^2-1)}$

$= \frac{1}{2}\left[\frac{n^2+n-2}{2n(n+1)}\right]$

$\frac{n^2+n-2}{4n(n+1)}$

Question Number	Scheme	Marks
(a)	$\frac{2}{r(r+1)(r-1)} = \frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}$	M1A1A1 (3)
(b)	$r=2 \quad 1 - \frac{2}{2} + \frac{1}{3}$	
	$r=3 \quad \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$	
	$r=4 \quad \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$	M1
	$r=n-1 \quad \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}$	
	$r=n \quad \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$	M1
	$\sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{2}{r} + \frac{1}{r+1}\right) = \left(1 - \frac{2}{2} + \frac{1}{2} + \frac{1}{n} - \frac{2}{n} + \frac{1}{n+1}\right)$	A1
	$\frac{1}{2}\sum_{r=1}^{n} \frac{2}{r(r+1)(r-1)} = \frac{1}{2} \times \left(\frac{1}{2} - \frac{1}{n} + \frac{1}{n+1}\right) = \frac{n^2+n-2}{4n(n+1)}$	dM1A1 (4)
		[7]

5. Evaluate the improper integral

$$\int_{2}^{\infty} 3e^{(4-2x)}\,dx$$

(3)

$$\lim_{t\to\infty}\int_2^t 3e^{4-2x}\,dx$$

$$=\lim_{t\to\infty}\left[\frac{3e^{4-2x}}{-2}\right]_2^t$$

$$=\lim_{t\to\infty}\frac{3e^{4-2t}}{-2}-\frac{3e^{0}}{-2}$$

$$=\frac{3}{2}$$

Question	Scheme	Marks	AOs
5(i)	$\int 3e^{(4-2x)}\,dx = -\frac{3}{2}e^{(4-2x)}$	B1	1.1b
	$\int_2^{\infty} 3e^{(4-2x)}\,dx = \lim_{t\to\infty}\left[\left(-\frac{3}{2}e^{(4-2t)}\right)-\left(-\frac{3}{2}e^{(4-2\times 2)}\right)\right]$	M1	2.1
	$=\frac{3}{2}$	A1	1.1b
		(3)	

(a) Write $\frac{4}{4x+1} - \frac{3}{3x+2}$ in the form $\frac{C}{(4x+1)(3x+2)}$, where C is a constant.

(1)

(b) Evaluate the improper integral

$$\int_1^{\infty} \frac{10}{(4x+1)(3x+2)} \mathrm{d}x$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant.

(6)

(Total 7 marks)

(a) Write $\frac{4}{4x+1} - \frac{3}{3x+2}$ in the form $\frac{C}{(4x+1)(3x+2)}$, where C is a constant.

(1)

(b) Evaluate the improper integral

$$\int_1^{\infty} \frac{10}{(4x+1)(3x+2)}\,dx$$

showing the limiting process used and giving your answer in the form $\ln k$, where k is a constant.

(6)

(Total 7 marks)

$4(3x+2) - 3(4x+1) = 5$

$C = 5$

b) $2\int \frac{4}{4x+1} - \frac{3}{3x+2}\,dx$

$\lim_{t\to\infty} \left[2\ln\left|\frac{4x+1}{3x+2}\right|\right]_1^t$

$2\ln\frac{4}{3} - ~~2\ln\left(\frac{5}{5}\right)~~ = \ln\frac{16}{9}$

(a) $\frac{12x+8-12x-3}{(4x+1)(3x+2)} = \frac{5}{(4x+1)(3x+2)}$

Accept $C = 5$

B1

1

(b) $\int \frac{10}{(4x+1)(3x+2)}\,dx = 2\int\left(\frac{4}{(4x+1)} - \frac{3}{(3x+2)}\right)dx$

M1

$= 2[\ln(4x+1) - \ln(3x+2)]\ (+c)$

OE

A1

$I = \lim_{a\to\infty} \int_1^a \left(\frac{10}{(4x+1)(3x+2)}\right)dx$

∞ replaced by a and $\lim_{a\to\infty}$ (OE)

M1

Limiting process shown.
Dependent on the previous M1M1

$= 2\lim_{a\to\infty}[\ln(4a+1) - \ln(3a+2)] - (\ln 5 - \ln 5)$

$= 2\lim_{a\to\infty}\left[\ln\left(\frac{4a+1}{3a+2}\right)\right] = 2\lim_{a\to\infty}\left[\ln\left(\frac{4+\frac{1}{a}}{3+\frac{2}{a}}\right)\right]$

$= 2\ln\frac{4}{3} = \ln\frac{16}{9}$

CSO

Q6.

Given that y = artanh(cos x)

(a) show that

$$\frac{dy}{dx} = -\operatorname{cosec} x$$

(2)

(b) Hence find the exact value of

$$\int_0^{\frac{\pi}{6}} \cos x \operatorname{artanh}(\cos x)\, dx$$

giving your answer in the form $a \ln\left(b + c\sqrt{3}\right) + d\pi,$ where a, b, c and d are rational numbers to be found.

(5)

(Total for question = 7 marks)

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a) $\tanh y = \cos x$

$\text{sech}^2 y \frac{dy}{dx} = -\sin x$

$\frac{dy}{dx} = \frac{-\sin x}{\text{sech}^2 y}$

$= \frac{-\sin x}{-\tanh^2 y + 1}$

$= \frac{-\sin x}{1 - \cos^2 x}$

$= \frac{-\sin x}{\sin^2 x} = \frac{-1}{\sin x}$

$= -\text{cosec}\, x$

$u = \text{artanh}(\cos x)$, $v = \sin x$

$u' = -\text{cosec}\, x$, $v' = \cos x$

$\left[\sin x \,\text{artanh}(\cos x)\right] - \int \sin x\left(-\frac{1}{\sin x}\right) dx$

$\left[\sin x \,\text{artanh}(\cos x) + x\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\text{artanh}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}$

$\frac{1}{4}\ln\left|\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right| + \frac{\pi}{6}$

$= \frac{1}{4}\ln\left|7+4\sqrt{3}\right| + \frac{\pi}{6}$

Question	Scheme		Marks
(a)	$y = \text{artanh}(\cos x)$		
	$\frac{dy}{dx} = \frac{1}{1-\cos^2 x} \times -\sin x$	Correct use of the chain rule	M1
	$= \frac{-\sin x}{\sin^2 x} = \frac{-1}{\sin x} = -\text{cosec}\, x$ *	A1: Correct completion with no errors	A1
			(2)
	Alternative 1		
	$\tanh y = \cos x \Rightarrow \text{sech}^2 y \frac{dy}{dx} = -\sin x$		
	$\frac{dy}{dx} = \frac{-\sin x}{\text{sech}^2 y} = \frac{-\sin x}{1-\cos^2 x}$	Correct differentiation to obtain a function of x	M1
	$= \frac{-\sin x}{\sin^2 x} = \frac{-1}{\sin x} = -\text{cosec}\, x$ *	A1: Correct completion with no errors	A1
			(2)
	Alternative 2		
	$\text{artanh}(\cos x) = \frac{1}{2}\ln\left(\frac{1+\cos x}{1-\cos x}\right)$		
	$\frac{dy}{dx} = \frac{1}{2} \times \frac{1-\cos x}{1+\cos x} \times \frac{-\sin x(1-\cos x) - \sin x(1+\cos x)}{(1-\cos x)^2}$	Correct differentiation to obtain a function of x	M1
	$= \frac{-2\sin x}{2(1-\cos^2 x)} = -\text{cosec}\, x$ *	A1: Correct completion with no errors	A1
			(2)

Question	Scheme		Marks
(b)	$\int \cos x \,\text{artanh}(\cos x)\,dx = \sin x \,\text{artanh}(\cos x) - \int \sin x \times -\text{cosec} x \,dx$ M1: Parts in the correct direction A1: Correct expression		M1 A1
	$\left[\sin x \,\text{artanh}(\cos x) + x\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\text{artanh}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}(-(0))$ M1: Correct use of limits on either part (provided both parts are integrated). Lower limit need not be shown		M1
	$= \frac{1}{4}\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right) + \frac{\pi}{6}$	Use of the logarithmic form of artanh	M1
	$= \frac{1}{4}\ln\left(7+4\sqrt{3}\right) + \frac{\pi}{6}$ or $\frac{1}{2}\ln\left(2+\sqrt{3}\right) + \frac{\pi}{6}$	Cao (oe)	A1
	The last 2 M marks may be gained in reverse order.		(5)
			(7 marks)

8. Water and antifreeze are being mixed together in a tank.

The mixture is stirred continuously so that the water and antifreeze are instantly dispersed evenly throughout the tank.

Initially the tank holds a mixture of 8 litres of water and 2 litres of antifreeze, so that the concentration of antifreeze in the mixture is said to be 20%.

The concentration of antifreeze in the mixture is now increased by

- adding water to the tank at a rate of 0.1 litres per second
- adding antifreeze to the tank at a rate of 0.3 litres per second
- pumping mixture from the tank at a rate of 0.4 litres per second.

Let x litres be the amount of antifreeze in the tank at time t seconds after the mixture starts to be altered.

(a) Show that the change in the amount of antifreeze in the tank can be modelled by the differential equation

$$\frac{dx}{dt} = 0.3 - \frac{x}{k}$$

where k is a positive constant to be determined.

(2)

(b) By solving the differential equation, determine how long it will take for the concentration of antifreeze in the mixture to reach 40%, according to the model.

Give your answer to the nearest tenth of a second.

(6)

As t becomes large, the concentration of antifreeze in the mixture approaches c%, where c is a constant.

(c) Find the value of c

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(Total for Question 8 is 10 marks)

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(Total for Question 8 is 10 marks)

$V = 8 + 2 + 0.1t + 0.3t - 0.4t$

$V = 10$

rate in : 0.3

rate out : $0.4 \times \frac{x}{10} = \frac{x}{25}$

$\therefore \frac{dx}{dt} = 0.3 - \frac{x}{25}$

$\dot{x} + \frac{1}{25}x = 0.3$

IF: $e^{\int \frac{1}{25} dt} = e^{\frac{1}{25}t}$

$\frac{d}{dt}\left(e^{\frac{1}{25}t}x\right) = 0.3e^{\frac{1}{25}t}$

$e^{\frac{1}{25}t}x = 7.5e^{\frac{1}{25}t} + c$

$x = 7.5 + Ce^{-\frac{1}{25}t}$

$t = 0,\ x = 2 \Rightarrow C = -5.5$

c) as $x \to 7.5$

$\frac{7.5}{10} = 75\%$

$\frac{\ln\left(\frac{3.5}{5.5}\right)}{-\frac{1}{25}} = t$

$t = 11.3$

Question	Scheme		Marks	AOs
8(a)	Volume of mixture = (8 + 2) litres therefore Rate of antifreeze out $= 0.4 \times \frac{x}{10}$ litres per second		M1	3.3
	$\frac{dx}{dt} = 0.3 - \frac{x}{25}$		A1	1.1b
			(2)	
(b)	Rearranges $\frac{dx}{dt} + \frac{x}{25} = 0.3$ and attempts integrating factor IF $= e^{\int \frac{1}{25} dt} = \ldots$	Separates the variables $\int \frac{1}{7.5 - x} dx = \int \frac{1}{25} dt$ $\Rightarrow \ldots$	M1	3.1a
	$xe^{\frac{t}{25}} = \int 0.3e^{\frac{t}{25}} dt \Rightarrow xe^{\frac{t}{25}} = \lambda e^{\frac{t}{25}}(+c)$	Integrates to the form $\lambda \ln(7.5 - x) = \frac{1}{25}t(+c)$	M1	1.1b
	$xe^{\frac{t}{25}} = 7.5e^{\frac{t}{25}} + c$	$-\ln(7.5 - x) = \frac{1}{25}t + c$	A1ft	1.1b
	$t = 0,\ x = 2 \Rightarrow c = \ldots$		M1	3.4
	$x = \frac{7.5e^{\frac{t}{25}} - 5.5}{e^{\frac{t}{25}}} = 4$ rearranges to achieve $e^{\frac{t}{5}} = \alpha$ and solves to find a value for t or $x = 7.5 - 5.5e^{-\frac{t}{25}} = 4$ rearranges to achieve $e^{-\frac{t}{25}} = \beta$ and solves to find a value for t	$-\ln(7.5 - 4) = \frac{1}{25}t - \ln 5.5$ Leading to a value for t	M1	3.4
	$t =$ awrt 11.3 seconds		A1	2.2b
			(6)	
(c)	$x = \frac{7.5e^{\frac{t}{25}} - 5.5}{e^{\frac{t}{25}}} = \frac{7.5 - 5.5e^{-\frac{t}{25}}}{1}$ so $= \lim_{t \to \infty} x = \frac{7.5 - 5.5 \times 0}{1} = 7.5$ Then 7.5 as a percentage of 10…		M11	3.5a
	$c = 75$ (%)		A1	2.2b
			(2)	
			(10 marks)	