13Fm Core Pure Maths Mock Paper 1 2021.05.14

This exam has 7 questions, for a total of 75 marks.

- Print in "booklets" will allow all questions to be on the left hand side.
- If instead you print in 2-in-1 settings, print the second page up to the last page first, then print the first page separately.

Question	Marks	Score
1	8	
2	10	
3	8	
4	11	
5	14	
6	11	
7	13	
Total:	75	

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Last updated: 22nd February 2023

1. (a) Find

$$\int \frac{1}{x^2 + 6x + 25}\,\mathrm{d}x$$

(3)

(b) Hence find the exact value of

$$\int_{-3}^{1} \left(1 - \frac{25}{x^2 + 6x + 25}\right)\,\mathrm{d}x$$

giving your answer in simplest form.

(3)

A student, Onisha, claims that the magnitude of the answer to part (b) gives the total area bounded by the curve $y = 1 - \dfrac{25}{x^2 + 6x + 25}$ and the x-axis between the line $x = -3$ and the line

$x = 1$

(c) State, with a reason, whether Onisha is correct.

(2)

Question 1 continued

Question 1 continued

Question 1 continued

(Total for Question 1 is 8 marks)

2. Jesse's company operate a coal mine. He is concerned about the mine running out of coal. He estimated that 2.5 million tonnes of coal are left in the mine, and he wishes to mine all of this coal in 20 years.

In order to mine the coal in a regulated manner, Jesse models the amount of coal to be mined in the coming years by the formula

$$M_r = \frac{10}{r^2 + 8r + 15}$$

where M_r is the amount of coal, in millions of tonnes, mined in year r, with the first year being year 1

(a) Show that, according to Jesse's model, the total amount of coal, in millions of tonnes, mined in the first n years is given by

$$T_n = \frac{9n^2 + 41n}{k(n+4)(n+5)}$$

where k is a constant to be determined.

(6)

(b) Explain why, according to Jesse's model, the mine will never run out of coal.

(2)

The company decides to mine an extra fixed amount each year so that all the coal will be mined in exactly 20 years.

(c) Refine the formula for M, so that 2.5 million tonnes of coal will be exhausted in exactly 20 years of mining.

(2)

Question 2 continued

Question 2 continued

(Total for Question 2 is 10 marks)

3. (a) Find, in terms of the real constant k, the determinant of the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & k & 2 \end{pmatrix}$$

(2)

Three distinct planes, Π_1, Π_2 and Π_3 are defined by the equations

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 4$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\Pi_3 : x + ky + 2z = -1$$

where λ and μ are scalar parameters.

(b) Find an equation in Cartesian form for

(i) Π_1

(ii) Π_2

(4)

Given that the three planes Π_1, Π_2 and Π_3 forms a sheaf,

(c) use the answer to part(a) to explain why $k = -1$

(2)

Question 3 continued

Question 3 continued

Question 3 continued

(Total for Question 3 is 8 marks)

4. (i) Prove by induction that, for $n \in \mathbb{N}$,

$$\begin{pmatrix} 4 & -1 \\ 9 & -2 \end{pmatrix}^n = \begin{pmatrix} 3n+1 & -n \\ 9n & 1-3n \end{pmatrix}$$

(6)

(ii) Consider the statement

$$n^2 < 2^n \qquad \text{for all } n \in \mathbb{Z}^+$$

A student, Hibah, attempts to prove this statement using induction as follows.

Hibah's response

Basis:
For $n = 1$ we have $1^2 = 1$ and $2^1 = 2$
Since $1 < 2$ the statement is true for $n = 1$
Assumption:
Suppose the statement is true for $n = k$, so $k^2 < 2^k$
Induction:

Line 1 $\mapsto$ $(k+1)^2 = k^2 + 2k + 1 < k^2 + k^2$ (since $2k + 1 < k^2$ for $k \in \mathbb{Z}^+$)

$$\begin{aligned} &= 2k^2 \\ &< 2 \times 2^k \qquad \text{(by the assumption } k^2 < 2^k\text{)} \\ &= 2^{k+1} \end{aligned}$$

Hence the result is true for $n = k + 1$
Conclusion:
the result is true of $n = 1$,
and if it is true for $n = k$ then it is true for $n = k + 1$,
therefore by mathematical induction the statement $n^2 < 2^n$ is true for all positive integers n.

(a) Show by a counterexample that the statement is not true

(1)

Given that the only mathematical error in Hibah's proof occurs in line 1 of the induction step,

(b) identify the error made in the student's proof,

(1)

(c) hence determine for which positive integers the statement is true, explaining your reasoning.

(3)

Question 4 continued

Question 4 continued

Question 4 continued

(Total for Question 4 is 11 marks)

5.

$$y = \arctan(\sinh(x))$$

(a) Show that $\dfrac{d^3y}{dx^3} = \dfrac{dy}{dx} - 2\left(\dfrac{dy}{dx}\right)^3$

(7)

(b) Hence find $\dfrac{d^5y}{dx^5}$ in terms of $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ and $\dfrac{d^3y}{dx^3}$

(4)

(c) Find the Maclaurin series for y, in ascending powers of x, up to and including the term in x^5

(3)

Question 5 continued

Question 5 continued

Question 5 continued

(Total for Question 5 is 14 marks)

6.

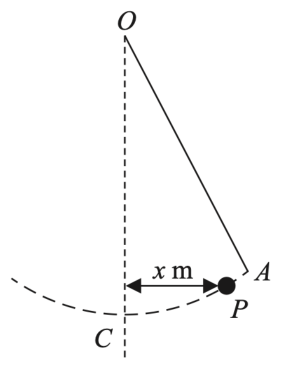

Figure 1

A child, Perpetual, plays on a rope swing.

Figure 1 represents Perpetual and the rope swing.

One end of the rope is attached to a tree and Perpetual sits on a large knot at the other end of the rope.

Perpetual swings back and forth in a vertical plane.

The rope is modelled as a light and inextensible string.

Perpetual is modelled as a particle at P. The rope is attached to the tree at the point O and the point C is vertically below O.

The horizontal displacement of P from the line OC at time t seconds ($t \neq 0$) is x metres.

The motion of P is modelled by the differential equation

$$\ddot{x} + 2\dot{x} + \lambda x = 0$$

where λ is a positive constant.

Perpetual is initially at rest, at the point A, with a horizontal displacement of 1.5 m from the line OC.

Given that the initial horizontal acceleration of Perpetual is -7.5 ms^{-2}

(a) show that $\lambda = 5$

(2)

Using the model,

(b) find an expression for the horizontal displacement of Perpetual at time t

(7)

Given that, when $t = 4.5$, Perpetual is vertically below O,

(c) evaluate the model, explaining your reasoning.

(2)

Question 6 continued

Question 6 continued

Question 6 continued

(Total for Question 6 is 11 marks)

7.

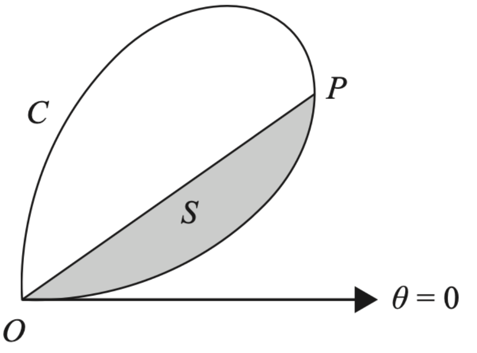

Figure 2

Figure 2 shows a sketch of curve C with polar equation

$$r = 3\sin(2\theta) \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

The point P on C has polar coordinates (R, ϕ).
The tangent to C at P is perpendicular to the initial line.

(a) Show that $\tan\phi = \frac{1}{\sqrt{2}}$

(4)

(b) Determine the exact value R

(2)

The region, S, shown shaded in Figure 2, is bounded by C and line the OP, where O is the pole.

(c) Use calculus to show that the exact area of S is

$$p\arctan\frac{1}{\sqrt{2}} + q\sqrt{2}$$

(7)

Question 7 continued

Question 7 continued

Question 7 continued

Question 7 continued

Question 7 continued

Question 7 continued

(Total for Question 7 is 13 marks)

Total for paper is 75 marks