12Fm October Exam 2022

Total 90 marks, 1 hour 45 minutes

Topic List

- Pure:
  1. Indices
  2. Quadratics
  3. Equations and Inequalities
  4. Graphs and Transformations
  5. Lines and Circles (Coordinate Geometry)
  6. Factor Theorem
  7. First half of Pure Chapter 11, Vectors.
- Stats:
  1. Probability
  2. Sampling (no LDS)
- Mechanics:
  1. Kinematics (with equations)
  2. Kinematics (with graphs)

Section A: Pure

1.

$$f(x) = 2x^3 - 7x^2 - 10x + 24$$

(a) Use the factor theorem to show that $(x + 2)$ is a factor of $f(x)$

(2)

(b) Hence factorise $f(x)$ completely

(3)

a) $f(-2)=0 \quad 2(-2)^3 - 7(-2)^2 - 10(-2) + 24 = 0$

$(x+2)$ is a factor.

---

way 1 $\quad (x+2)(ax^2+bx+c) = 2x^3 - 7x^2 - 10x + 24$

$ax^3 + bx^2 + cx$

$+ 2ax^2 + 2bx + 2c =$

Math Deg Norm2 d/c a+bi

$aX^3 + bX^2 + cX + d = 0$

X1	4
X2	1.5
X3	-2

REPEAT

4

$a=2$	$2a+b=-7$	$c+2b=-10$	$2c=24$
	$4+b=-7$	$c-22=-10$	$c=12$
	$b=-11$	$c=12$	

$(x+2)(2x^2-11x+12)$

$2x^3 - 7x^2 - 10x + 24 = (x+2)(2x-3)(x-4)$

$y=0$

$x=4 \quad x=\frac{3}{2} \quad x=-2$

$(x-4)(2x-3)(x+2)$

$2(x-4)(x-1.5)(x+2)$

way2

$$\begin{array}{r} 2x^2 - 11x + 12 \\ x+2 \overline{\big)\, 2x^3 - 7x^2 - 10x + 24} \\ 2x^3 + 4x^2 \quad\quad\quad\quad \\ \hline -11x^2 - 10x \quad\quad \\ -11x^2 - 22x \quad\quad \\ \hline 12x + 24 \\ 12x + 24 \\ \hline 0 \end{array}$$

$$\Rightarrow (x+2)(2x^2-11x+12)$$

$$= (x+2)(2x-3)(x-4)$$

In this question you should show all stages of your working.
Solutions relying on calculator technology are not acceptable.

2.

(a) Solve

$$3^x = \frac{1}{27} = \frac{1}{3^3} = 3^{-3}$$

(2)

(b)

$$\frac{2^{x+1}}{4^{y-2}} = 64$$

Express y in terms of x, writing your answer in simplest form

(3)

a)

$$3^x = 3^{-3}$$

$$x = -3$$

① $(a^b)^c = a^{bc}$

② $a^{b-c} = \frac{a^b}{a^c}$

b)

$$\frac{2^{x+1}}{(2^2)^{y-2}} = \frac{2^{x+1}}{2^{2y-4}} = 2^{(x+1)-(2y-4)} = 2^6$$

$$x+1-2y+4 = 6$$

$$x-2y = 1$$

$$2y = x-1$$

$$y = \frac{x-1}{2} \; //$$

3.

(a) On separate axes sketch the graphs of

(i) $y = -3x + c$ where c is a positive constant $\rightarrow c > 0$

(ii) $y = \dfrac{1}{x} + 5$

On each sketch show the coordinates of any point at which the graph crosses the y-axis and the equation of any horizontal asymptote.

(4)

Given that the two curves in part (a) meet at two distinct points,

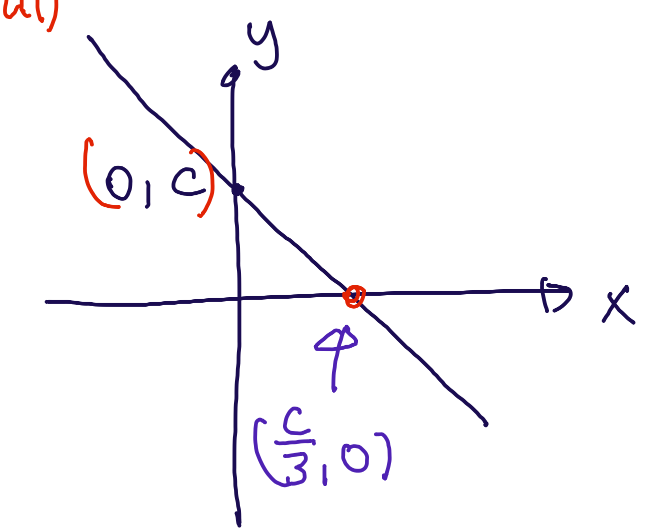
(b) Show that $(5 - c)^2 > 12$

(3)

(c) **Hence** find the set of possible values for c.

(4)

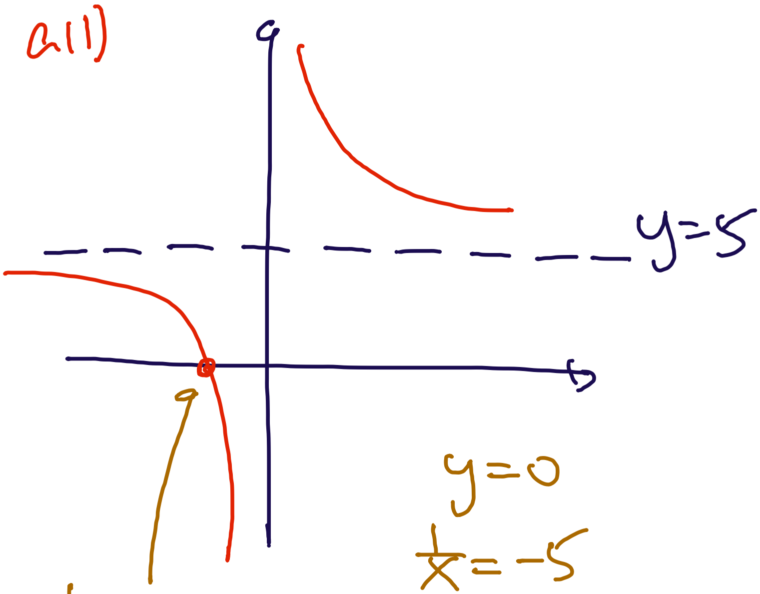
a i)



$y = 0$, $3x = c$

$x = \frac{c}{3}$

a ii)



$\left(\frac{-1}{5}, 0\right)$

$y = 0$

$\frac{1}{x} = -5$

$x = \frac{-1}{5}$

(b) they meet at two points $b^2 - 4ac > 0$

$$-3x + c = \frac{1}{x} + 5$$

$$-3x^2 + cx = 1 + 5x$$

$$3x^2 + x(5-c) + 1 = 0$$

$b^2 - 4ac > 0$

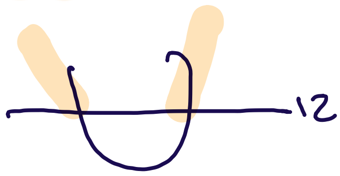
$(5-c)^2 - 4(3)(1) > 0$

$(5-c)^2 - 12 > 0$

as required

(c) **Hence** find the set of possible values for c.

Way 1



$\boxed{c > 0}$

$(5-c)^2 > 12$

$(5-c) > \sqrt{12}$ OR $(5-c) < -\sqrt{12}$

$5 - c > \sqrt{12}$ $\quad -c + 5 < -\sqrt{12}$

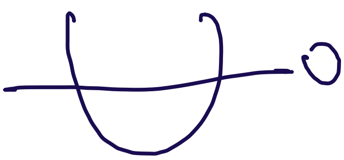
$c < 5 - \sqrt{12}$ $\quad 5 + \sqrt{12} < c$

$c > 5 + \sqrt{12}$

$0 \quad 5-\sqrt{12} \quad 5+\sqrt{12}$

$\{0 < c < 5 - \sqrt{12}\} \cup \{c > 5 + \sqrt{12}\}$

Way 2



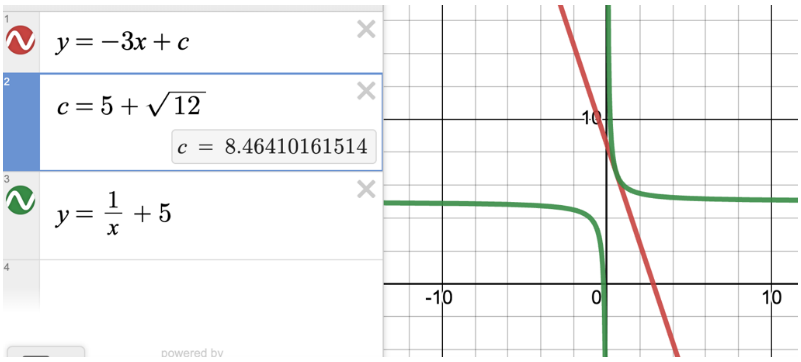
$c^2 - 10c + 25 - 12 > 0$

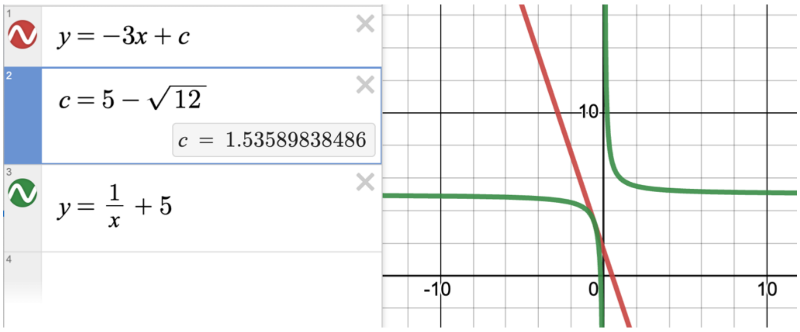
$c^2 - 10c + 13 > 0$

$c > 5 + 2\sqrt{3}$ $\quad c < 5 - 2\sqrt{3}$

$\{0 < c < 5 - \sqrt{12}\} \cup \{c > 5 + \sqrt{12}\}$

https://www.desmos.com/calculator/puparfp1ej ⇐ click for desmos "slider"





4.

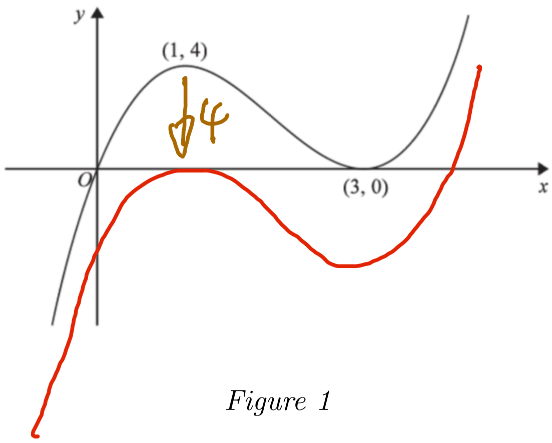


Figure 1

Figure 1 shows a sketch of the curve with equation $y = f(x)$ where

$$f(x) = x(3-x)^2 \qquad\qquad x \in \mathbb{R}$$

The curve passes through the origin and touches the x-axis at the point $(3, 0)$. There is a maximum point at $(1, 4)$ and a minimum point at $(3, 0)$.

(a) On separate diagrams, sketch the curve with equation

(i) $y = f\left(\frac{1}{2}x\right)$

(ii) $y = f(x+2)$

On each sketch indicate clearly the coordinates of

- any points where the curve crosses or touches the x-axis,
- any points where the curve crosses or touches the y-axis,
- any maximum or minimum points.

ii) $(x+2)(3-(x+2))^2$

$x=0$

$y = 2(3-2)^2 = 2$

$k = -4$

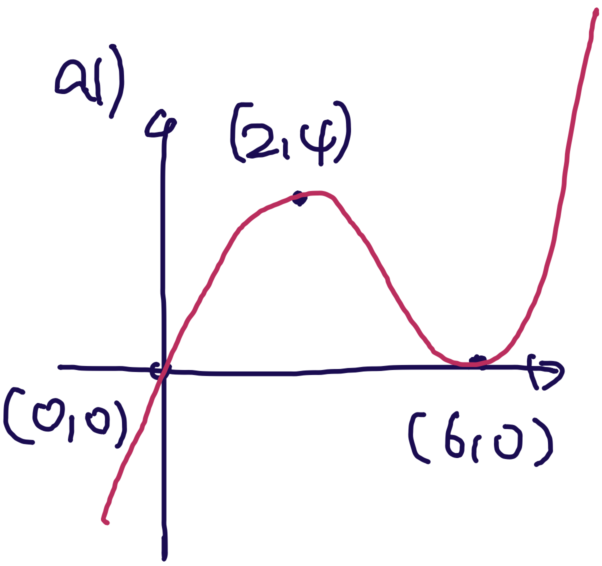
b) $a = 1$

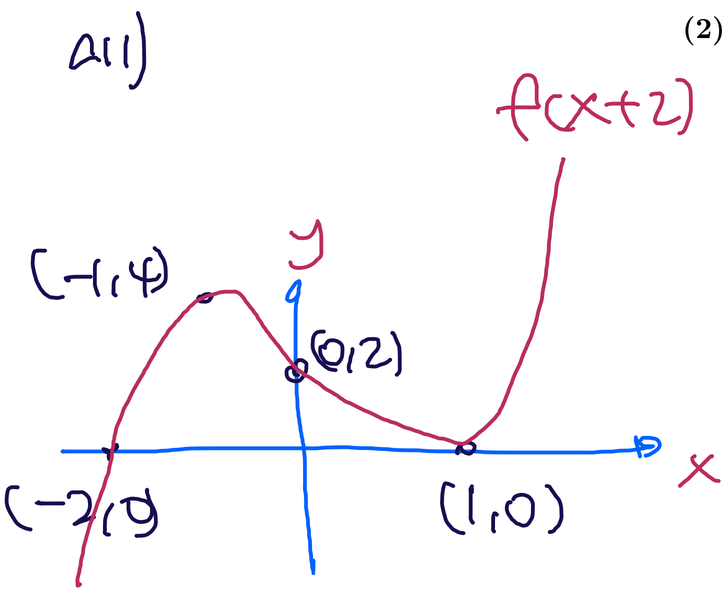
(6)

Given that the curve with equation $y = f(x) + k$, where k is a non-zero constant, has a maximum point at $(a, 0)$.

(b) State the values of a and k

(2)





5.

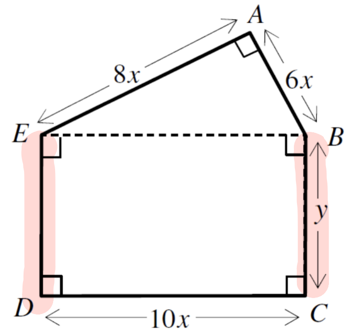


Figure 2

Figure 2 shows a pentagon $ABCDE$ whose measurements, in cm, are in terms of x and y.

(a) If the perimeter of the pentagon is 120 cm, shows that its area A cm^2 is given by

$$A = 600x - 96x^2$$

(4)

(b) Determine the maximum value for the area of the pentagon and the corresponding value of x which produces this maximum area.

(5)

$P = 120$ $\quad 8x + 6x + 2y + 10x = 120$

$2y = 120 - 24x$

$y = 60 - 12x$

$\text{Area} = 10xy + \frac{1}{2}(8x)(6x)$

$= 10x(60 - 12x) + 24x^2$

$= 600x - 120x^2 + 24x^2$

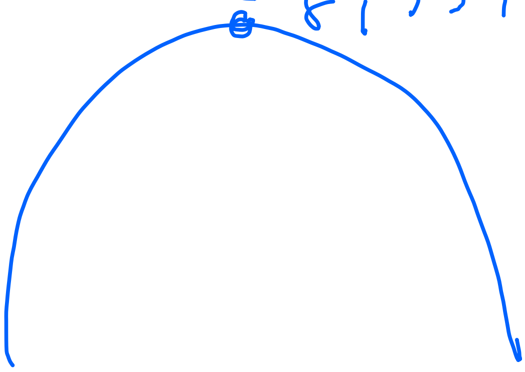
$= 600x - 96x^2$ as required

$A = -96\left(x^2 - \frac{25}{4}x\right)$

$$A = -96\left[\left(x - \frac{25}{8}\right)^2 - \frac{625}{64}\right]$$

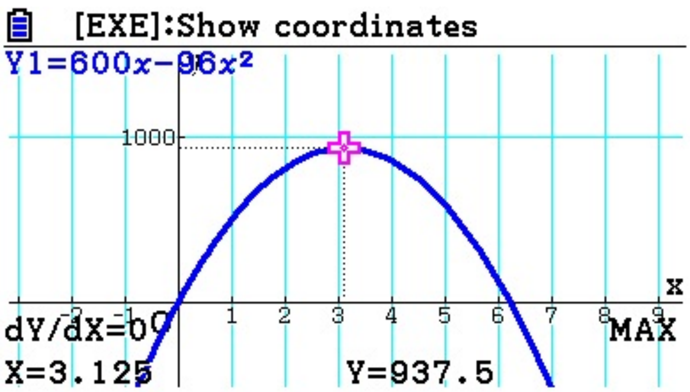
$$A = -96\left(x - \frac{25}{8}\right)^2 + 937.5$$

$$\left(\frac{25}{8}, 937.5\right)$$



$$\begin{cases} x = \frac{25}{8} \\ A = 937.5 \end{cases}$$

(b)



$$A = 600x - 96x^2$$

$$\frac{dA}{dx} = 600 - 192x$$

$$\frac{dA}{dx} = 0 \qquad 192x = 600$$

$$x = 3.125$$

$$A = 600(3.125) - 96(3.125)^2$$

$$A = 937.5$$ //

6.

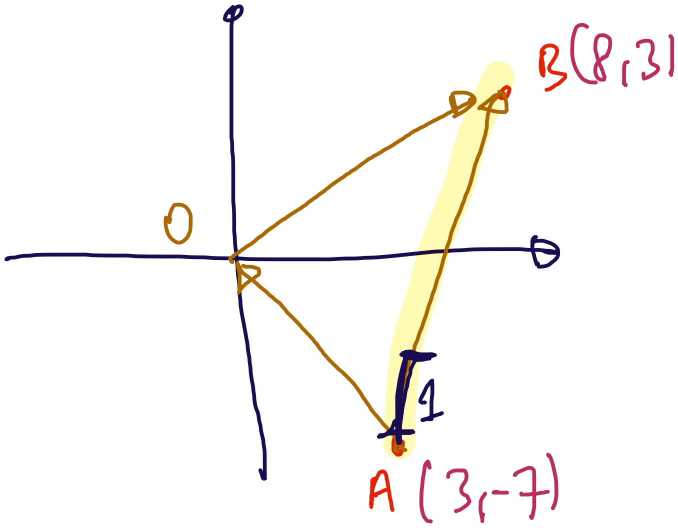
Given that the point A has position vector $3\mathbf{i} - 7\mathbf{j}$ and the point B has position vector $8\mathbf{i} + 3\mathbf{j}$.

(a) Find $\overrightarrow{AB}$ **(2)**

(b) Find $|\overrightarrow{AB}|$ **(2)**

(c) Find a unit vector in the direction of $\overrightarrow{AB}$ **(2)**

(d) Find, as a bearing, the direction of $\overrightarrow{AB}$ **(2)**



a)

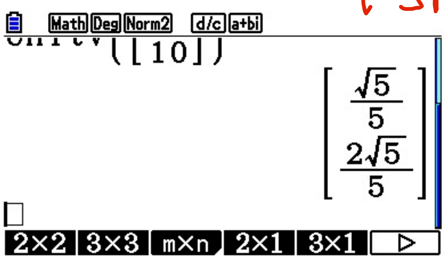
$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$

$$= \begin{pmatrix} -3 \\ 7 \end{pmatrix} + \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

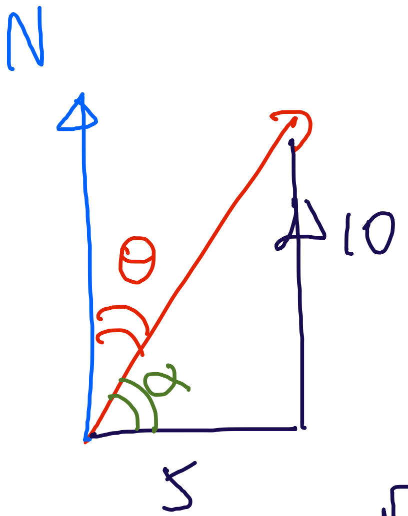
$$= \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

b) $\sqrt{5^2 + 10^2} = 5\sqrt{5}$

c) $\frac{1}{5\sqrt{5}} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} \frac{5}{5\sqrt{5}} \\ \frac{10}{5\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$



$= \begin{pmatrix} \frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \end{pmatrix}$



$\tan \alpha = \frac{10}{5}$

$\alpha = 63°$

$\theta = 90 - 63$

$\boxed{\text{Bearing } 027°}$

7.

The circle C has centre $X(3,5)$ and radius r.
The line l has equation $y = 2x + k$, where k is a constant.

(a) Show that l and C intersect when

$$5x^2 + (4k - 26)x + k^2 - 10k + 34 - r^2 = 0$$

(3)

Given that l is a tangent to C,

(b) Show that $5r^2 = (k+p)^2$, where p is a constant to be found.

(3)

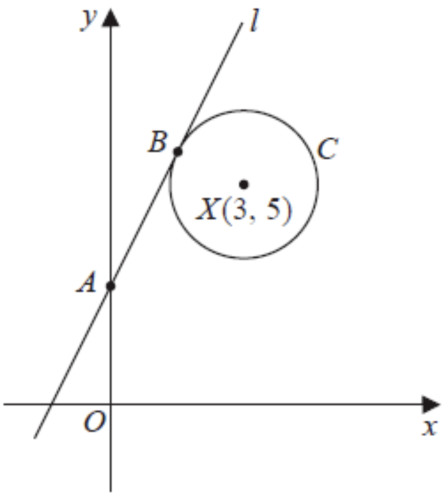


Figure 3

Figure 3 shows the line l which

- cuts the y-axis at the point A
- and touches the circle C at the point B

Given that $AB = 2r$

(c) find the value of k.

(6)

$(x-3)^2+(y-5)^2=r^2$ —①

$y=2x+k$ —②

a) $x^2-6x+9+y^2-10y+25=r^2$

$(a+b)^2$
$= a^2+2ab+b^2$

sub ② into eqt

$x^2-6x+9+(2x+k)^2$
$-10(2x+k)+25=r^2$

$x^2-6x+9+4x^2+4kx+k^2-20x-10k+25-r^2=0$

$$5x^2+(4k-26)x+k^2-10k+34-r^2=0$$

b) Given that l is a tangent to C,

(b) Show that $5r^2=(k+p)^2$, where p is a constant to be found. **(3)**

meet once ⇒ $\Delta=0$ $\quad b^2-4ac=0$

$(4k-26)^2-4(5)\left[k^2-10k+34-r^2\right]=0$

$16k^2-208k+676-20k^2+200k-680+20r^2=0$

$-4k^2-8k-4+20r^2=0$

$20r^2=4k^2+8k+4$

$k^2+2k+1=5r^2$

$5r^2=(k+1)^2$ $\quad\boxed{p=1}$

7.

The circle C has centre $X(3, 5)$ and radius r.
The line l has equation $y = 2x + k$, where k is a constant.

(a) Show that l and C intersect when

$$5x^2 + (4k - 26)x + k^2 - 10k + 34 - r^2 = 0$$

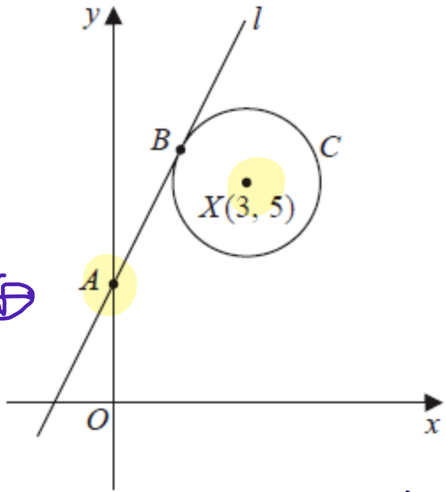
(3)

Given that l is a tangent to C, $5r^2 = (k+1)^2$

(b) Show that $5r^2 = (k + p)^2$, where p is a constant to be found.

(3)

$(0, k)$



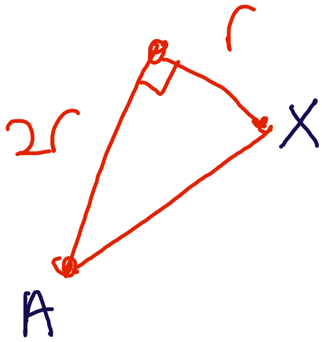


Figure 3

$$|AX| = \sqrt{(2r)^2 + (r^2)}$$
$$= \sqrt{4r^2 + r^2}$$
$$= \sqrt{5r^2}$$

Figure 3 shows the line l which

- cuts the y-axis at the point A
- and touches the circle C at the point B

Given that $AB = 2r$

(c) find the value of k.

(6)

$$|AX| = \sqrt{(3-0)^2 + (5-k)^2}$$
$$= \sqrt{9 + 25 - 10k + k^2}$$
$$= \sqrt{k^2 - 10k + 34}$$
$$\sqrt{5r^2} = \sqrt{k^2 - 10k + 34}$$
$$5r^2 = k^2 - 10k + 34$$

$$k^2-10k+34=k^2+2k+1$$

$$33=12k$$

$$k=\frac{11}{4}$$

Section B: Mechanics

8.

area

Two small children, Ajaz and Beth, are running a 100m race along a straight horizontal track.

- They both start from rest, leaving the start line at the same time.
- Ajaz accelerates at 0.8 ms^{-2} up to a speed of 4 ms^{-1} and then maintains this speed until he crosses the finish line.
- Beth accelerates at 1 ms^{-2} for T seconds and then maintains a constant speed until she crosses the finish line.
- Ajaz and Beth cross the finish line at the same time.

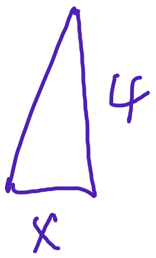
(a) Sketch on the same axes, a speed-time graph for each child, from the instant when they leave the start line to the instant when they cross the finish line. **(3)**

(b) Find the time taken by Ajaz to complete the race. 27.5 seconds // **(4)**

(c) Find the value of T 3.915 // **(4)**

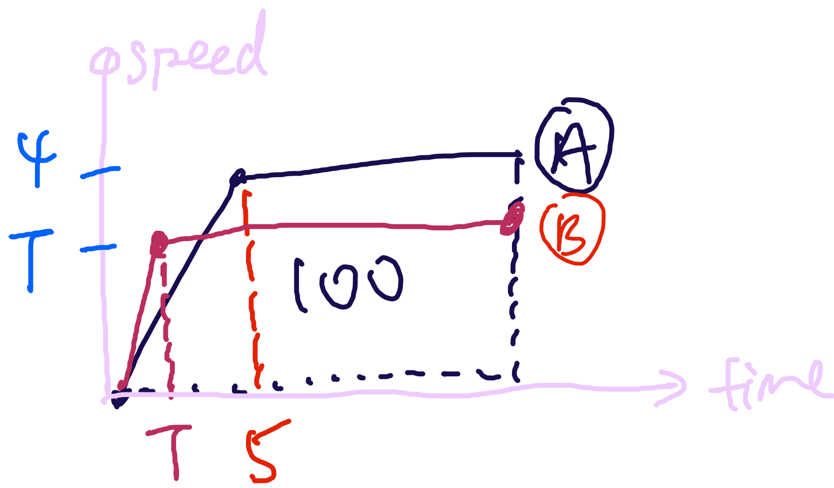
(d) Find the difference in the speeds of the two children as they cross the finish line. **(2)**

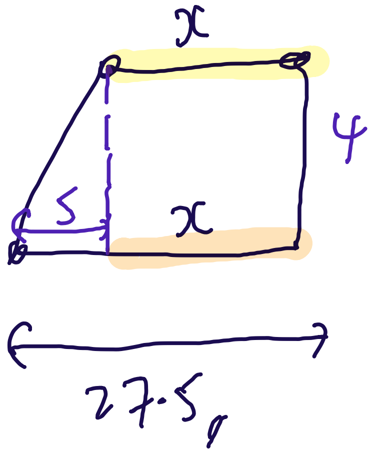
0.085 ms^{-1} //



$\frac{4}{x} = 0.8$

$x = 5$





area=100

$$\left(\frac{x+x+5}{2}\right)4 = 100$$

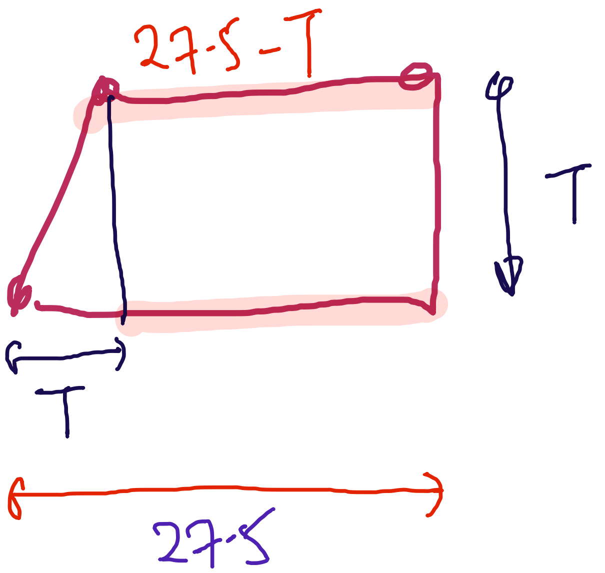
$$(2x+5)(2) = 100$$

$$2x+5 = 50$$

$$2x = 45$$

$$x = 22.5$$

the time needed = 27.5



$\boxed{T<27.5}$

area=100

$$\left(\frac{27.5+(27.5-T)}{2}\right)T = 100$$

$$(55-T)T = 200$$

$$55T - T^2 = 200$$

$$T^2 - 55T + 200 = 0$$

$T = 3.915$ only $\quad T = 51.084$ (rej)

ⓓ $4 - T = 0.085 \text{ ms}^{-1}$

9.

At time $t = 0$, a particle is projected vertically upwards with speed u from a point A. The particle moves freely under gravity. At time T the particle is at its maximum height H above A.

(a) Find T in terms of u and g **(2)**

(b) Show that $H = \dfrac{u^2}{2g}$ **(2)**

The point A is at a height $3H$ above the ground.

(c) Find, in terms of T, the total time from the instant of projection to the instant when the particle hits the ground. **(4)**

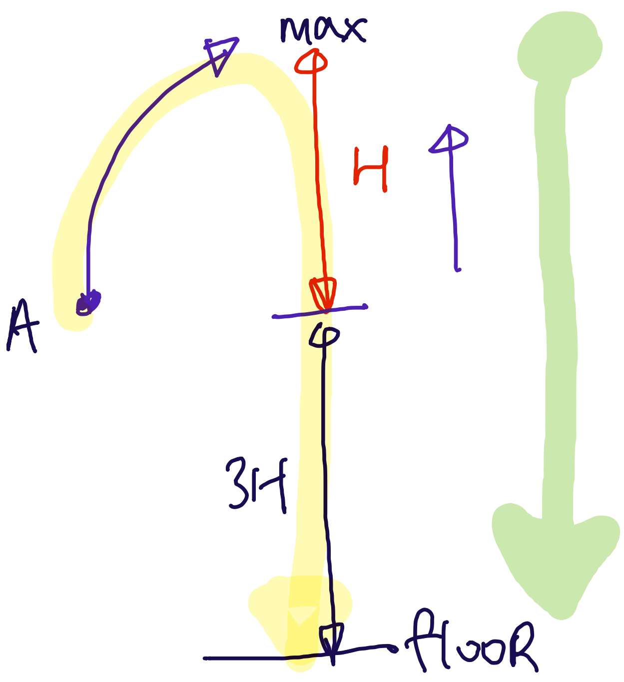
↑ +ve

$s = +H$
$u = +u$
$v = 0$
$a = -g$
$t = ?$

a)
$v = u + at$
$0 = u - gT$
$u = gT$
$T = \dfrac{u}{g}$

(b)
$v^2 - u^2 = 2as$
$0 - u^2 = 2(-g)(H)$
$u^2 = 2gH$
$H = \dfrac{u^2}{2g}$

ⓒ way 1



$\uparrow +ve$

$S = -3H$
$u = +u$
$a = -g$
$t = ?$

$$S = ut + \frac{1}{2}at^2$$

$$-3H = ut - \frac{g}{2}t^2$$

$$\frac{g}{2}t^2 - ut - 3H = 0$$

$$\frac{g}{2}t^2 - ut - \frac{3u^2}{2g} = 0$$

$$t = \frac{u \pm \sqrt{u^2 - 4\left(\frac{g}{2}\right)\left(\frac{-3u^2}{2g}\right)}}{g}$$

$$t = \frac{u \pm \sqrt{u^2 + 3u^2}}{g} = \frac{u \pm 2u}{g}$$

$t > 0$

$$t = \frac{3u}{g} = 3T$$

consider way 2 (from max to floor)

$s = +4H$ $\qquad \downarrow$ +ve

$u = 0$

$a = +g$

$t = ?$

$$s = ut + \frac{1}{2}at^2$$

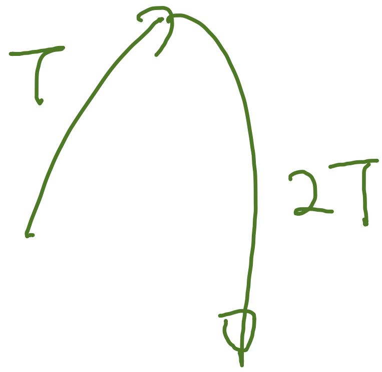
$$4H = \frac{g}{2}t^2$$

$$4\left(\frac{u^2}{2g}\right) = \frac{g}{2}t^2$$

$$\frac{2u^2}{g} = \frac{g}{2}t^2$$

$$\frac{4u^2}{g^2} = t^2$$

$$t = \frac{2u}{g}$$



total time 3T

Section C: Statistics

10.

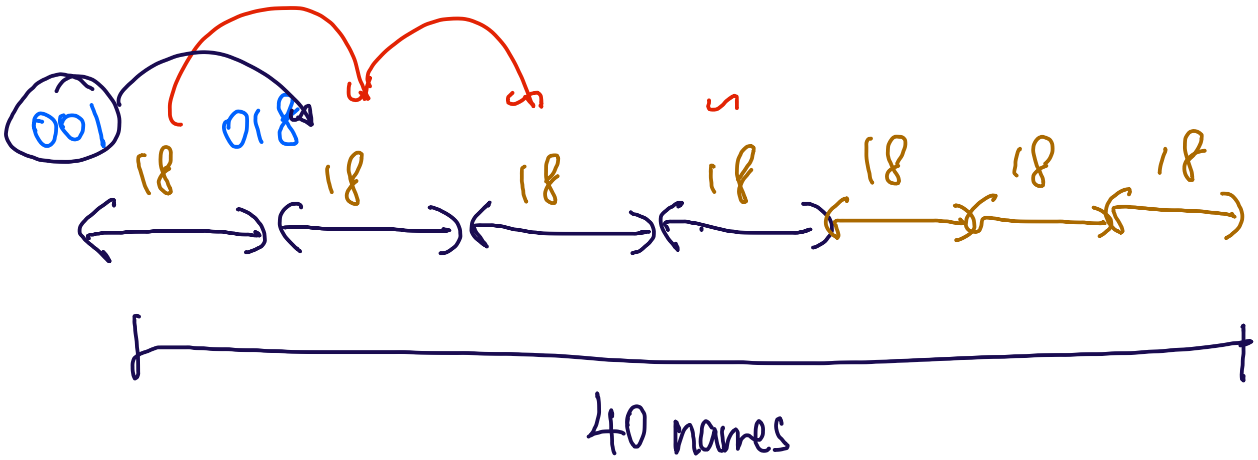
The names of the 720 members of a swimming club are listed alphabetically in the club's membership book. The chairman of the swimming club wishes to select a systematic sample of 40 names. The names are numbered from 001 to 720 and a number between 001 and w is selected at random. The corresponding name and every x^{th} name thereafter are included in the sample.

(a) Find the value of w. 018 **(1)**

(b) Find the value of x. 18 **(1)**

(c) Write down the probability that the sample includes both the first name and the third name in the club's membership book. 0 **(1)**

(d) State one advantage and one disadvantage of systematic sampling in this case. **(2)**



(d) quick/easy/simple to carry out. (advantage)

Some names eg first and third can never be picked together (disadvantage)

(d)	**Advantage:** Any one of: • Simple or easy to use also allow "quick" or "efficient" (o.e.) • It is suitable for large samples (or populations) • Gives a good spread of the data	B1
	Disadvantage: Any one of: • The alphabetical list is (probably) not random • Biased since the list is not (truly) random • Some combinations of names are not possible	B1
		(2)
		(5 marks)

Notes:

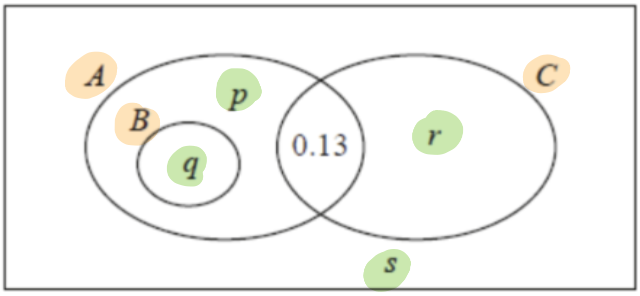
(d) If no labels are given treat the 1st reason as an advantage and the 2nd as a disadvantage

B1: For advantage

B1: For disadvantage – "it requires a sampling frame" is 2nd B0 since the alphabetical list is given.
Note: Do not score both B1 marks for opposing advantages and disadvantages.

11.

In the Venn diagram below, A, B, C are events and p, q, r and s are probabilities.
The events A and C are independent and $P(A) = 0.65$



(a) State which two of the events A, B, C are mutually exclusive. **(1)**

B , C

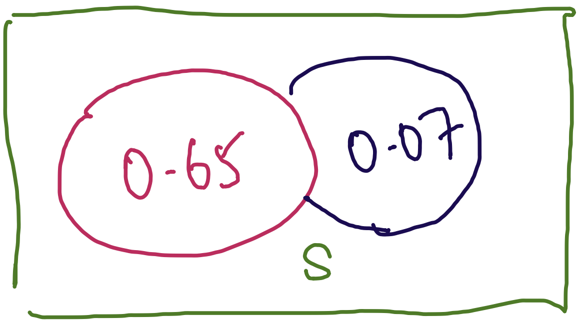
(b) Find the value of r and the value of s **(5)**

b)

$p(A) \times p(C) = p(A \cap C)$

$0.65 \cdot (0.13 + r) = 0.13$

$r = 0.07$ //



$\Sigma p = 1$

$0.65 + 0.07 + S = 1$

$S = 0.28$ //