Please check the examination details below before entering your candidate information

Pearson Edexcel Level 3 GCE

Centre Number: 1 3 2 0 2

Candidate Number:

Monday 14 March 2022

Morning (Time: 1 hour 30 minutes) | Paper Reference **9FM0/3A**

Further Mathematics

Advanced

Paper 3A: Further Pure Mathematics 1

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the box** at the top of this page with your candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided
  – there may be more space than you need.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 8 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets
  – use this as a guide as to how much time to spend on each question.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over ▶

1. An ellipse has equation

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

The point P lies on the ellipse and has coordinates $(5\cos\theta, 2\sin\theta)$, $0 < \theta < \frac{\pi}{2}$

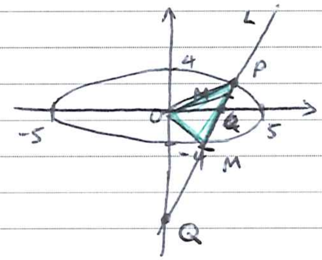
The line L is a normal to the ellipse at the point P.

An equation for L is

$$5x\sin\theta - 2y\cos\theta = 21\sin\theta\cos\theta$$

Given that the line L crosses the y-axis at the point Q and that M is the midpoint of PQ, find the exact area of triangle OPM, where O is the origin, giving your answer as a multiple of $\sin 2\theta$

(6)



$L: \ 5x\sin\theta - 2y\cos\theta = 21\sin\theta\cos\theta$

at Q: $x = 0$

$-2y\cos\theta = 21\sin\theta\cos\theta$

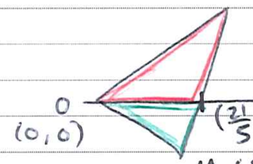
$\cos\theta(21\sin\theta + 2y) = 0$

$y = \frac{-21\sin\theta}{2}$

$P: (5\cos\theta, 2\sin\theta)$

$Q: \left(0, \frac{-21\sin\theta}{2}\right)$

$M: \left(\frac{5\cos\theta}{2}, \frac{-17\sin\theta}{4}\right)$



L cuts x-axis when $y = 0$ $\quad 5x\sin\theta = 21\sin\theta\cos\theta$

i.e. $x = \frac{21}{5}\cos\theta$

Area $\triangle OPM = \frac{1}{2}\times\left(\frac{21}{5}\cos\theta \times 2\sin\theta\right) + \frac{1}{2}\times\left(\frac{21}{5}\cos\theta \times \frac{17}{4}\sin\theta\right)$

$= \frac{21}{5}\sin\theta\cos\theta + \frac{357}{40}\sin\theta\cos\theta = \frac{105}{8}\sin\theta\cos\theta$

$= \boxed{\frac{105}{16}\sin 2\theta}$

Question 1 continued

(Total for Question 1 is 6 marks)

Turn over

2. The plane Π has equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + s\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

The point A has coordinates $(4, -3, c)$

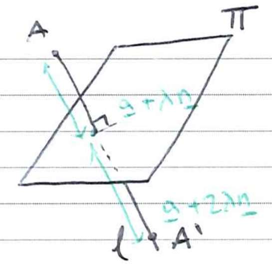
(a) Find, in terms of c, an equation for the line that passes through the point A and is perpendicular to the plane Π. Give your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

(3)

The point A' is a reflection of the point A in the plane Π

(b) Find, in terms of c, the coordinates of the point A'

(5)



(a) $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -3 \\ 2 & -1 & 4 \end{vmatrix}$

$= \underline{i}(8-3) - \underline{j}(4+6) + \underline{k}(-1-4)$

$= 5\underline{i} - 10\underline{j} - 5\underline{k}$

$l: \left(\underline{r} - \begin{pmatrix} 4 \\ -3 \\ c \end{pmatrix}\right) \times \begin{pmatrix} 5 \\ -10 \\ -5 \end{pmatrix} = \underline{0}$

simplify vector.

(b) Line l: $\underline{r} = \begin{pmatrix} 4 \\ -3 \\ c \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

Plane Π: $\underline{n} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ $\underline{r} \cdot \underline{n} = d$

$\begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 3+4+4 = 11$

$\underline{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 11$

Question 2 continued

Finding where l and Π intersect:

$l: \quad \underline{r} = \begin{pmatrix} 4+\lambda \\ -3-2\lambda \\ c-\lambda \end{pmatrix}$

$\rightarrow \Pi: \quad \begin{pmatrix} 4+\lambda \\ -3-2\lambda \\ c-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 11$

$4+\lambda+6+4\lambda-c+\lambda = 11$

$10-c+6\lambda = 11 \quad \Rightarrow \quad \lambda = \frac{c+1}{6}$

$A' \quad \underline{a}' = \underline{a} + 2\lambda$

$\underline{a}' = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ c \end{pmatrix} + \frac{c+1}{3}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 4+\frac{c+1}{3} \\ -3-2\left(\frac{c+1}{3}\right) \\ c-\left(\frac{c+1}{3}\right) \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{13+c}{3} \\ \frac{-11-2c}{3} \\ \frac{2c-1}{3} \end{pmatrix}$

A' has coordinates:

$\left(\frac{13+c}{3}, \frac{-11-2c}{3}, \frac{2c-1}{3}\right)$

(Total for Question 2 is 8 marks)

3. Given that $y = \cot x$,

(a) show that

$$\frac{d^2y}{dx^2} = 2\cot x + 2\cot^3 x$$

(3)

(b) Hence show that

$$\frac{d^3y}{dx^3} = p\cot^4 x + q\cot^2 x + r$$

where p, q and r are integers to be found.

(3)

(c) Find the Taylor series expansion of $\cot x$ in ascending powers of $\left(x - \frac{\pi}{3}\right)$ up to and including the term in $\left(x - \frac{\pi}{3}\right)^3$

(3)

(a) $y = \cot x$

$\frac{dy}{dx} = -\text{cosec}^2 x \quad \Rightarrow \quad \frac{d^2y}{dx^2} = -2\text{cosec}x\left(-\text{cosec}x\cot x\right)$

$\frac{d^2y}{dx^2} = 2\text{cosec}^2 x \cot x$

$= 2\left(1+\cot^2 x\right)\cot x$

$= 2\cot x + 2\cot^3 x \quad$ as req

(b) $\frac{d^3y}{dx^3} = -2\text{cosec}^2 x + 6\cot^2 x\left(-\text{cosec}^2 x\right)$

$= -2\left(1+\cot^2 x\right) + 6\cot^2 x\left(-1-\cot^2 x\right)$

$= -2 - 2\cot^2 x - 6\cot^2 x - 6\cot^4 x$

$= -6\cot^4 x - 8\cot^2 x - 2$

$p = -6$
$q = -8$
$r = -2$

Question 3 continued

(c) $$y \approx y\left(\frac{\pi}{3}\right) + \left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}}\left(x-\frac{\pi}{3}\right) + \left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{3}}\frac{\left(x-\frac{\pi}{3}\right)^2}{2}$$

$$+ \left.\frac{d^3y}{dx^3}\right|_{x=\frac{\pi}{3}}\frac{\left(x-\frac{\pi}{3}\right)^3}{6} + \dots$$

$$y\left(\frac{\pi}{3}\right) = \cot\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{3} \qquad \frac{d^2y}{dx^2}\left(\frac{\pi}{3}\right) = \frac{8}{3\sqrt{3}} = \frac{8\sqrt{3}}{9}$$

$$\frac{dy}{dx}\left(\frac{\pi}{3}\right) = \frac{-1}{\sin^2\left(\frac{\pi}{3}\right)} = \frac{-4}{3} \qquad \frac{d^3y}{dx^3}\left(\frac{\pi}{3}\right) = \frac{-16}{3}$$

$$y = \frac{\sqrt{3}}{3} - \frac{4}{3}\left(x-\frac{\pi}{3}\right) + \frac{8\sqrt{3}}{9}\left(\frac{1}{2}\right)\left(x-\frac{\pi}{3}\right)^2 - \frac{16}{3}\left(\frac{1}{6}\right)\left(x-\frac{\pi}{3}\right)^3.$$

$$\cot(x) = \frac{\sqrt{3}}{3} - \frac{4}{3}\left(x-\frac{\pi}{3}\right) + \frac{4\sqrt{3}}{3}\left(x-\frac{\pi}{3}\right)^2 - \frac{8}{9}\left(x-\frac{\pi}{3}\right)^3$$

Question 3 continued

Question 3 continued

(Total for Question 3 is 9 marks)

4. Given that

$$\frac{7-2x}{|5x-3|+1} < 4$$

(a) show that

$$|5x-3| > \frac{3-2x}{4}$$

(2)

(b) On the same diagram, sketch the graph with equation $y = |5x-3|$ and the graph with equation $y = \dfrac{3-2x}{4}$ showing the coordinates of the points where the graphs meet the coordinate axes.

(3)

(c) Use algebra to determine the exact set of values of x for which

$$\frac{7-2x}{|5x-3|+1} < 4$$

giving your answer in set notation.

(4)

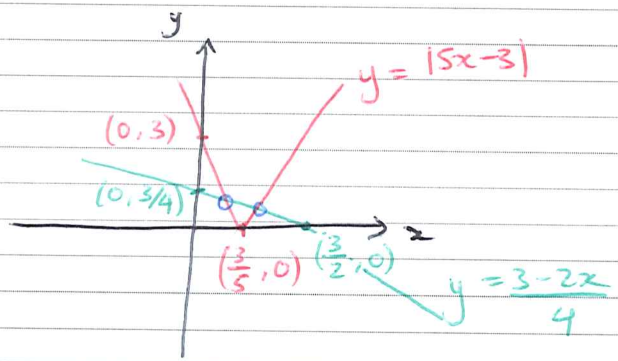
(a) $7-2x < 4(|5x-3|+1)$

$\Rightarrow 7-2x < 4|5x-3| + 4$

$\Rightarrow 3-2x < 4|5x-3|$

$\Rightarrow |5x-3| > \dfrac{3-2x}{4}$

(b)



(c) $y = |5x-3|$ above $y = \dfrac{3-2x}{4}$

Question 4 continued

'left' branch: $y = 3 - 5x = \frac{3-2x}{4}$

$x < \frac{3}{5}$

$12 - 20x = 3 - 2x$

$18x = 9$

$x = \frac{1}{2}$ $\qquad x < \frac{1}{2}$

'Right' branch:

$x \geq \frac{3}{5}$ $\qquad 5x - 3 = \frac{3-2x}{4}$

$20x - 12 = 3 - 2x$

$22x = 15$

$x = \frac{15}{22}$ $\qquad x > \frac{15}{22}$

Soln: $\{x \in \mathbb{R} : x < \frac{1}{2}\} \cup \{x \in \mathbb{R} : x > \frac{15}{22}\}$

Question 4 continued

Question 4 continued

(Total for Question 4 is 9 marks)

5. With respect to a fixed origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = 18\mathbf{i} - 14\mathbf{j} - 2\mathbf{k} \quad \overrightarrow{OB} = -7\mathbf{i} - 5\mathbf{j} + 3\mathbf{k} \quad \overrightarrow{OC} = -2\mathbf{i} - 9\mathbf{j} - 6\mathbf{k}$$

The points O, A, B and C form the vertices of a tetrahedron.

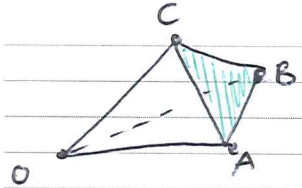
(a) Show that the area of the triangular face ABC of the tetrahedron is 108 to 3 significant figures.
(3)

(b) Find the volume of the tetrahedron.
(4)

An oak wood block is made in the shape of the tetrahedron, with centimetres taken for the units.

The density of oak is $0.85\,\mathrm{g\,cm^{-3}}$

(c) Determine the mass of the block, giving your answer in kg.
(2)



(a) Area $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$\overrightarrow{AB} = \begin{pmatrix} -7 \\ -5 \\ 3 \end{pmatrix} - \begin{pmatrix} 18 \\ -14 \\ -2 \end{pmatrix} = \begin{pmatrix} -25 \\ 9 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -2 \\ -9 \\ -6 \end{pmatrix} - \begin{pmatrix} 18 \\ -14 \\ -2 \end{pmatrix} = \begin{pmatrix} -20 \\ 5 \\ -4 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -25 & 9 & 5 \\ -20 & 5 & -4 \end{vmatrix} = \underline{i}(-36-25) - \underline{j}(100+100) + \underline{k}(-125+180)$$

$$= -61\underline{i} - 200\underline{j} - 55\underline{k}$$

$$\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left(\sqrt{61^2 + 200^2 + 55^2}\right)$$

$$= 108.104\ldots$$

$$= 108 \ (3\text{ s.f.})$$

Question 5 continued

(b) $V = \frac{1}{6}\left|\vec{OA} \cdot (\vec{OB} \times \vec{OC})\right|$

$= \frac{1}{6}\begin{vmatrix} 18 & -14 & -2 \\ -7 & -5 & 3 \\ -2 & -9 & -6 \end{vmatrix}$

$= \frac{1}{6}\left[18(30+27) + 14(42+6) - 2(63-10)\right]$

$= \frac{1592}{6} \quad = \frac{796}{3}$

$\boxed{V = \frac{796}{3}}$

(c) $m = \text{vol} \times \text{density}$

$= \frac{796}{3} \times 0.85$

$= 225.533\ldots$ g

$= 0.226$ kg (3sf)

Question 5 continued

Question 5 continued

(Total for Question 5 is 9 marks)

6. On a particular day, the depth of water in a river estuary at a specific location is modelled by the equation

$$D = 2\sin\left(\frac{x}{3}\right) + 3\cos\left(\frac{x}{3}\right) + 6 \qquad 0 \leqslant x \leqslant 7\pi \qquad \text{(I)}$$

where the depth of water is D metres at time x hours after midnight on that day.

(a) Write down the depth of water at midnight, according to the model.

(1)

Using the substitution $t = \tan\left(\frac{x}{6}\right)$

(b) show that equation (I) can be re-written as

$$D = \frac{3t^2 + 4t + 9}{1 + t^2}$$

(3)

(c) Hence determine, according to the model, the time after midnight when the depth of water is 5 metres for the first time. Give your answer to the nearest minute.

(5)

(a) $D = 2\sin\left(\frac{x}{3}\right) + 3\cos\left(\frac{x}{3}\right) + 6$

$x = 0$ $\quad D = 3\cos(0) + 6$

$= 9$ $\quad$ 9m

(b) $t = \tan\left(\frac{x}{6}\right)$ $\quad \sin\left(\frac{x}{3}\right) = \frac{2t}{1+t^2}$ $\quad \cos\left(\frac{x}{3}\right) = \frac{1-t^2}{1+t^2}$

$D = 2\left(\frac{2t}{1+t^2}\right) + 3\left(\frac{1-t^2}{1+t^2}\right) + 6$

$= \frac{4t + 3 - 3t^2 + 6 + 6t^2}{1+t^2}$

$D = \frac{3t^2 + 4t + 9}{1+t^2}$ $\quad$ as req

Question 6 continued

(a) $D=5$ $\qquad 5 = \dfrac{3t^2+4t+9}{1+t^2}$

$\Rightarrow \quad 5+5t^2 = 3t^2+4t+9$

$\Rightarrow \quad 2t^2-4t-4=0$

$\Rightarrow \quad t^2-2t-2=0$

$t = \dfrac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$

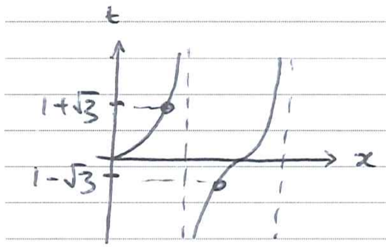
$t = \tan\left(\frac{x}{6}\right)$

$\arctan(t) = \dfrac{x}{6}$ $\qquad x = 6\arctan(1+\sqrt{3})$ gives earliest time

$= 7.319\ldots$ hrs

7 hrs 19 mins

Time 7:19 am



Question 6 continued

Question 6 continued

(Total for Question 6 is 9 marks)

7. The curve H has equation

$$xy = a^2 \qquad x > 0$$

where a is a positive constant.

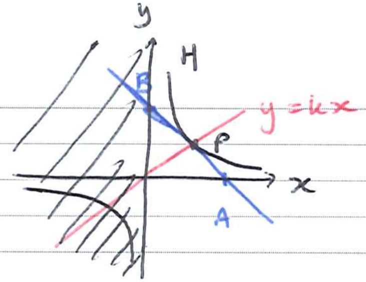
The line with equation $y = kx$, where k is a positive constant, intersects H at the point P

(a) Use calculus to determine, in terms of a and k, an equation for the tangent to H at P **(4)**

The tangent to H at P meets the x-axis at the point A and meets the y-axis at the point B

(b) Determine the coordinates of A and the coordinates of B, giving your answers in terms of a and k **(2)**

(c) Hence show that the area of triangle AOB, where O is the origin, is independent of k **(2)**



(a) P: $x(kx) = a^2$

$kx^2 = a^2$

$x = \frac{a}{\sqrt{k}}$

$y = a\sqrt{k}$

$P\left(\frac{a}{\sqrt{k}}, a\sqrt{k}\right)$

$xy = a^2$

$y + x\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{-y}{x}$

at P $\qquad \frac{dy}{dx} = \frac{-\cancel{a}\sqrt{k}}{\cancel{a}/\sqrt{k}} = -k$

$y - a\sqrt{k} = -k\left(x - \frac{a}{\sqrt{k}}\right)$

$\Rightarrow \boxed{y = -kx + 2a\sqrt{k}}$

Question 7 continued

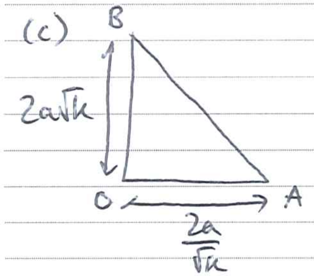
(b) $y = -kx + 2a\sqrt{k}$

A: $0 = -kx + 2a\sqrt{k}$

$x = \dfrac{2a\sqrt{k}}{k} = \dfrac{2a}{\sqrt{k}}$ A: $\left(\dfrac{2a}{\sqrt{k}}, 0\right)$

B: $y = 2a\sqrt{k}$ B: $(0, 2a\sqrt{k})$.

(c)



Area $\Delta AOB = \frac{1}{2} \times \dfrac{2a}{\sqrt{k}} \times 2a\sqrt{k}$

$= 2a^2$

which is independent of k.

Question 7 continued

Question 7 continued

(Total for Question 7 is 8 marks)

Turn over

8. A community is concerned about the rising level of pollutant in its local pond and applies a chemical treatment to stop the increase of pollutant.

The concentration, x parts per million (ppm), of the pollutant in the pond water t days after the chemical treatment was applied, is modelled by the differential equation

$$\frac{dx}{dt} = \frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3}x \tanh t \qquad \text{(I)}$$

When the chemical treatment was applied the concentration of pollutant was 3 ppm.

(a) Use the iteration formula

$$\left(\frac{dy}{dx}\right)_n \approx \frac{(y_{n+1} - y_n)}{h}$$

once to estimate the concentration of the pollutant in the pond water 6 hours after the chemical treatment was applied.

(4)

(b) Show that the transformation $u = x^3$ transforms the differential equation (I) into the differential equation

$$\frac{du}{dt} + u \tanh t = 1 + \frac{3}{\cosh t} \qquad \text{(II)}$$

(3)

(c) Determine the general solution of equation (II)

(4)

(d) Hence find an equation for the concentration of pollutant in the pond water t days after the chemical treatment was applied.

(3)

(e) Find the percentage error of the estimate found in part (a) compared to the value predicted by the model, stating if it is an overestimate or an underestimate. **(3)**

(a) $\frac{dx}{dt} = \frac{3 + \cosh t}{3x^2 \cosh t} - \frac{1}{3}x \tanh t$

$t = 0, \quad x = 3$ $\qquad$ $6 \text{ hrs} = \frac{6}{24} \text{ days} = 0.25 \text{ days}.$

$t = 0.25, \quad x = ?$

$\left(\frac{dx}{dt}\right)_0 \approx \frac{x_1 - x_0}{h}$ $\qquad$ $x_1 \approx x_0 + h\left(\frac{dx}{dt}\right)_0$

Question 8 continued

$$x_1 \approx 3 + 0.25\left(\frac{3+\cosh 0}{3\times 3^2\cosh 0} - \frac{1}{3}(3)\tanh(0)\right)$$

$$\approx 3 + 0.25\left(\frac{4}{27}\right)$$

$$\approx \frac{82}{27} \text{ ppm}$$

$x_1 \approx 3.04$ ppm after 6 hrs.

(b) $u = x^3$ $\qquad \frac{dx}{dt} = \frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t$

$$\frac{du}{dx} = 3x^2$$

$$\frac{du}{dt} = \frac{du}{dx} \times \frac{dx}{dt}$$

$$\frac{du}{dt} = 3x^2\left(\frac{3+\cosh t}{3x^2\cosh t} - \frac{1}{3}x\tanh t\right)$$

$$= \frac{3+\cosh t}{\cosh t} - x^3\tanh t$$

$$\Rightarrow \frac{du}{dt} = \frac{3}{\cosh t} + 1 - u\tanh t$$

$$\Rightarrow \frac{du}{dt} + u\tanh t = \frac{3}{\cosh t} + 1 \quad \text{(II)}$$

Question 8 continued

(c) $\text{I.F}: e^{\int \tanh t \, dt} = e^{\ln\cosh t} = \cosh t$

$$\cosh t \frac{du}{dt} + u\sinh t = 3 + \cosh t$$

$$\frac{d}{dt}(u\cosh t) = 3 + \cosh t$$

$$u\cosh t = \int 3 + \cosh t \, dt$$

$$= 3t + \sinh t + C$$

G.S $\quad u = \frac{3t}{\cosh t} + \tanh t + \frac{C}{\cosh t}$

where C is a constant

(d) $u = x^3$ $\qquad t=0,\ x=3,\ u=27$

$$27 = 0 + 0 + \frac{c}{1} \Rightarrow c = 27$$

$$x^3 = \frac{3t+27}{\cosh t} + \tanh t$$

$$\Rightarrow x = \left(\frac{3t+27}{\cosh t} + \tanh t\right)^{1/3}$$

Question 8 continued

(e) $t = 0.25$ $x = \left(\tanh(0.25) + \frac{3(0.25)+27}{\cosh(0.25)} \right)^{\frac{1}{3}}$

$= 3.0055\ldots$

% error $\left(\left| \frac{3.0055\ldots - \left(\frac{82}{27}\right)}{3.0055\ldots} \right| \right) \times 100$

$= 1.048\ldots$

% error $= 1.05\%$ (3 sf)

Question 8 continued

(Total for Question 8 is 17 marks)

TOTAL FOR PAPER IS 75 MARKS