# Year 13 Mathematics Mock Set#04d
# Pure Paper 1
# Year 13 revision session 2023

- Advised to print in "A3-booklets", this will allow all questions to be on the left hand side.
- You can also print in A4, double-sided, and two staples on the left
- If instead you print in 2-in-1 settings, first print the second page up to the last page, then print the cover page separately (to allow all questions on the left)

This exam paper has 14 questions, for a total of 100 marks.

| Question | Marks | Score |
|---|---|---|
| 1 | 6 | |
| 2 | 5 | |
| 3 | 6 | |
| 4 | 7 | |
| 5 | 4 | |
| 6 | 9 | |
| 7 | 9 | |
| 8 | 6 | |
| 9 | 6 | |
| 10 | 8 | |
| 11 | 7 | |
| 12 | 10 | |
| 13 | 6 | |
| 14 | 11 | |
| Total: | 100 | |

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Last updated: 12th May 2023

**1.** Mr Chan reports the number of visits of SoHokMaths at the end of each month from September 2022. He believes that the relationship between the number of visits, $n$, and the number of months after the start of the academic year, $t$, can be modelled by $n = a \times 2^{kt}$ where $a$ and $k$ are constants.

(a) Show that, according to the model, the graph of $\log_{10} n$ against $t$ is a straight line.

**(2)**

(b) Figure 1 shows a plot of the values of $t$ and $\log_{10} n$ from September 2022, the point at $t = 1$, to March 2023, the point at $t = 7$.

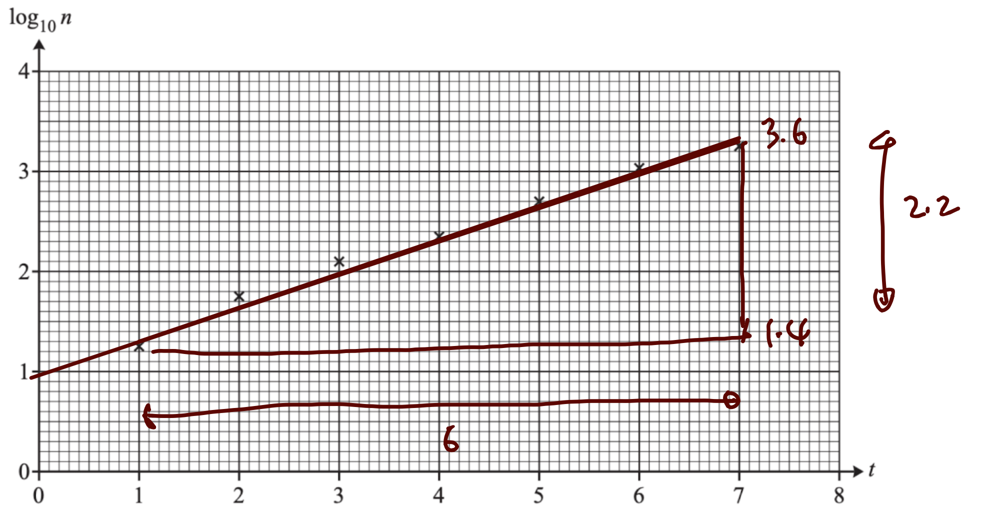



**Figure 1**

Find estimates of the values of $a$ and $k$.

**(4)**

a) $n = a \times 2^{kt}$

$\log n = \log(a \cdot 2^{kt})$

$= \log a + \log 2^{kt}$

$= \log a + kt(\log 2)$

$= \log a + k\log 2 \; t$

b)

gradient $\simeq \frac{2.2}{6} \simeq 0.37$

intercept $\simeq 0.9$

$\log n = 0.37t + 0.9$

$\log a = 0.9$, $k\log 2 \simeq 0.37$

$a = 10^{0.9}$, $k \simeq 1.22$

$\Rightarrow a = 7.94 \quad k \simeq 1.22$

$n = 7.94 \times 2^{1.22t}$ //

**2.** Given that $\mathbf{v} = 2\mathbf{a} + 3\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are position vectors $\mathbf{a} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$.

(a) Determine the magnitude of $\mathbf{v}$.

**(3)**

(b) Determine the angle between $\mathbf{v}$ and the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

**(2)**

a)

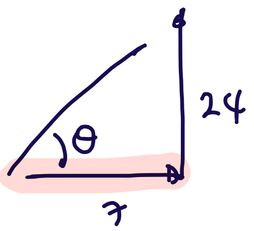
$$V = 2\begin{pmatrix} 5 \\ 3 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

$$V = \begin{pmatrix} 10 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ 18 \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$$

$$|V| = \sqrt{7^2 + 24^2} = 25 \; //$$

b)

$$V = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$$




$$\tan\theta = \frac{24}{7}$$

$$\theta = 73.7^\circ \; (3sf) \; //$$

**3.** (a) Prove the identity

$$\frac{\sin^3\theta}{\sin\theta - 1} - \frac{\sin^2\theta}{1+\sin\theta} \equiv -\tan^2\theta(1+\sin^2\theta)$$

(4)

(b) Hence solve the equation

$$\frac{\sin^3\theta}{\sin\theta - 1} - \frac{\sin^2\theta}{1+\sin\theta} = \tan^2\theta(1-\sin^2\theta)$$

for $0 < \theta < 2\pi$

(2)

a)

$$\frac{\sin^3\theta + \sin^4\theta - \sin^3\theta + \sin^2\theta}{(\sin^2\theta - 1)}$$

$$= \frac{\sin^2\theta(\sin^2\theta+1)}{-\cos^2\theta} = -\tan^2\theta(\sin^2\theta+1)$$

$$= \text{RHS} //$$

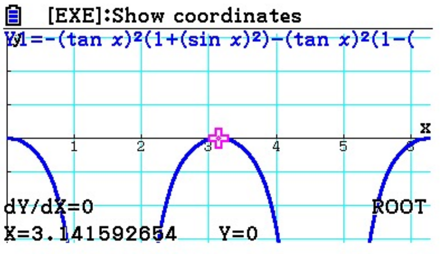
b)

$$-\tan^2\theta(1+\sin^2\theta) = \tan^2\theta(1-\sin^2\theta)$$

$$\tan^2\theta[-1-\sin^2\theta-1+\sin^2\theta] = 0$$

$\tan\theta = 0$ or $-1-\sin^2\theta = 1-\sin^2\theta$

$\theta = \pi //$ (no solution)



**4.**

$$\theta = \arcsin x$$

(a) By expressing $\arccos x$ in terms of $\theta$, show that

$$\arcsin x + \arccos x = \frac{\pi}{2}$$

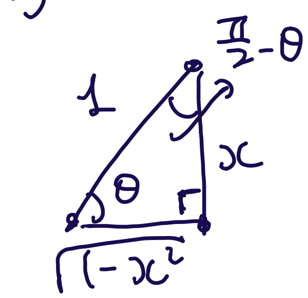
**(3)**

(b) Hence, or otherwise, solve the equation

$$3 \arcsin (y - 1) = 2 \arccos (y - 1)$$

**(4)**

a)




$\sin\theta = x$

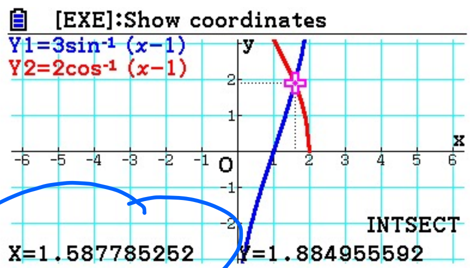
$\cos(\frac{\pi}{2} - \theta) = x$ [identity]

$\arccos x = \frac{\pi}{2} - \theta$

$\arcsin x + \arccos x$

$= \theta + \frac{\pi}{2} - \theta$

$= \frac{\pi}{2}$ as req.




check!

b) Let $y - 1 = u$

then $3\arcsin(u) = 2\arccos u$

$3\arcsin u = 2\left[\frac{\pi}{2} - \arcsin u\right]$

$3\arcsin u = \pi - 2\arcsin u$

$5\arcsin u = \pi$

$\arcsin u = \frac{\pi}{5}$

$u = \sin(\frac{\pi}{5})$

$y = u + 1$

$y = 1.59$ (2dp)

**5.** Four students, Tom, Andrew, Zakiyyah, Maria are attempting to complete the indefinite integral

$$\int \frac{1}{x}\,\mathrm{d}x \qquad \{x > 0\}$$

Each of the students' solutions is shown below:

| | |
|---|---|
| Tom | $\int \frac{1}{x}\,\mathrm{d}x = \ln x$ |
| Andrew | $\int \frac{1}{x}\,\mathrm{d}x = k\ln x$ |
| Zakiyyah | $\int \frac{1}{x}\,\mathrm{d}x = \ln Ax$ |
| Maria | $\int \frac{1}{x}\,\mathrm{d}x = \ln x + c$ |

(a) Tom and Andrew are both incorrect.

(i) Explain what is wrong with Tom's answer.

(ii) Explain what is wrong with Andrew's answer.

(2)

(b) Zakiyyah and Maria are both correct, explain why their answers are equivalent.

(2)

a i) Tom forgot to $+C$

ii) $K\ln x = \ln x^K$ which is not a $+C$

b) $\ln AX = \ln A + \ln x \Rightarrow$ that's $\ln A = C$ //

6. A diameter of a circle $C_1$ has end-points at $(-3,-5)$ and $(7,3)$.

(a) Find an equation of the circle $C_1$.

**(3)**

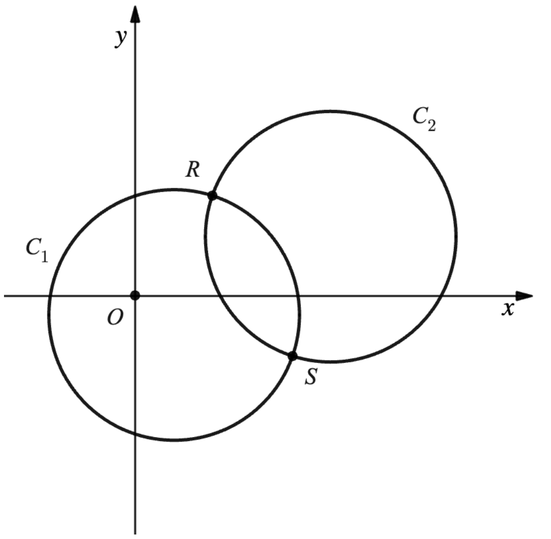



**Figure 2: https://www.desmos.com/calculator/a0zitxwiir**

The circle $C_1$ is translated by $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ to give circle $C_2$, as shown in Figure 3.

(b) Find an equation of the circle $C_2$.

**(2)**

The two circles intersect at points $R$ and $S$.

(c) Show that the equation of the line $RS$ is $y = -2x + 13$.

**(4)**

a) mid point = centre $= \left(\frac{-3+7}{2}, \frac{-5+3}{2}\right)$

$= (2, -1)$

radius $= \sqrt{(7-2)^2 + (3--1)^2} = \sqrt{41}$

check $\sqrt{(-3-2)^2 + (-5--1)^2} = \sqrt{41}$

$C_1$: $(x-2)^2 + (y+1)^2 = 41$ //

Question 6 continued

b) new centre: $(10, 3)$

$C_2$: $(x-10)^2+(y-3)^2=41$

c) either equate two circles,

or find mid-pt of centres, then ⊥.

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way 1 $x^2-4x+4+y^2+2y+1=x^2-20x+100+y^2-6y+9$

$16x+8y=104$

$y=-2x+13$

---

way 2 $(10,3)$ vs $(2,-1)$

$m=\frac{3+1}{10-2}=\frac{4}{8}$

$m_{\perp}=-2$

mid-pt $=\left(\frac{10+2}{2}, \frac{3-1}{2}\right)$

$=(6,1)$

$\Rightarrow y=-2x+c \Rightarrow y=-2x+13$ //

**7.** Given that

$$f(x) = 2x + 3 \qquad \text{for } x < 0$$
$$g(x) = x^2 - 2x + 1 \qquad \text{for } x > 1$$

(a) Find $gf(x)$, stating the domain. **(3)**

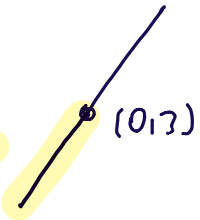
(b) State the range of $gf(x)$. **(1)**

(c) Find $(gf)^{-1}(x)$

Note here that even though question did not specify, you always need to state the domain for any functions you "found" **(5)**

7a) $f(x) = 2x+3$ for $x<0$

C $\Rightarrow f(x) < 3$




$g[f(x)] = (2x+3)^2 - 2(2x+3) + 1$

$= 4x^2 + 12x + 9 - 4x - 6 + 1$

$= 4x^2 + 8x + 4$

$= 4(x^2 + 2x + 1)$

$= 4(x+1)^2$

domain Ⓐ Ⓑ $A \cap B \cap C$

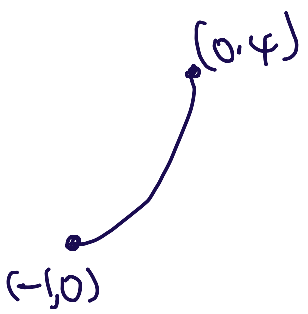
$\Rightarrow 2x+3 > 1$ and $x < 0$

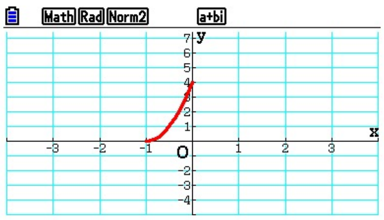
$x > -2$

$x > -1$

So $\boxed{-1 < x < 0}$

b)




range:

$0 < gf < 4$ //

**Question 7 continued**

c)

$$y = 4(x+1)^2$$

$$\frac{y}{4} = (x+1)^2$$

$$\pm\sqrt{\frac{y}{4}} = (x+1) \Rightarrow x = -1 \pm \sqrt{\frac{y}{4}}$$

but $-1 < x < 0$

therefore

$$f^{-1}(x) = -1 + \sqrt{\frac{x}{4}} \;//$$

domain $0 < x < 4$ //

**8.** A student of Mr Chan, Diogo, is proving that

$$n^3 - n \text{ is a multiple of 3 for all positive integer values of } n.$$

Diogo begins a proof by exhaustion.

Diogo's response

| | | |
|---|---|---|
| Step 1 | | $n^3 - n = n(n^2 - 1)$ |
| Step 2 | When $n = 3m$, where $m$ is a non-negative integer | $n^3 - n = 3m(9m^2 - 1)$ which is a multiple of 3 |
| Step 3 | When $n = 3m + 1$, | $n^3 - n = (3m+1)\big((3m+1)^2 - 1\big)$ |
| Step 4 | | $= (3m+1)(9m^2)$ <br> $= 3(3m+1)(3m^2)$ <br> which is a multiple of 3 |
| Step 5 | Therefore $n^3 - n$ is a multiple of 3 for all positive integer values of $n$. | |

(a) Explain the two mistakes that Diogo has made after Step 3.

**(2)**

(b) Correct Diogo's argument from Step 4 onwards.

**(4)**

a) 1 $(3m+1)[9m^2+6m]$ not $(3m+1)[9m^2]$

2 multiple of 3 $\Rightarrow$ need $(3m+2)$ too.

b) $(3m+1)(3)(3m^2+2m)$

$\Rightarrow$ which is a multiple of 3

$(3m+2)((3m+2)^2-1)$

$= (3m+2)[9m^2+12m+3]$

$= (3m+2)[3][3m^2+4m+1]$

$\Rightarrow$ which is a multiple of 3

Conclusion $\Rightarrow$ Step 5 Therefore $n^3 - n$ is a multiple of 3 for all positive integer values of $n$.

**9.** **Note that for Edexcel questions, (i) and (ii) means two unrelated questions, they are combined into one because they belong to the same chapter/topic.**

(i) The first four terms of a sequence are 2, 3, 0, 3 and the subsequent terms are given by $a_{k+4} = a_k$.

(a) State what type of sequence this is.

(1)

(b) Find $\sum_{k=1}^{200} a_k$.

(1)

(ii) A different sequence is given by $u_n = b^n$ where $b$ is a constant and $n \geq 1$.

(a) State the set of values of $b$ for which this is a divergent sequence. → nth term ↳ doesn't converge.

(2)

(b) In the case where $b = \frac{1}{3}$, find the sum of all the terms in the sequence.

(2)

i) a) periodic with order 4.

b) 200 terms $\Rightarrow \frac{200}{4} = 50$

$50[2+3+0+3] = 400$ //

NB!

ii) a) $b > 1$ or $b \leq -1$ — if $b=1$ it stays $1^n = 1$ so it still converges to 1.

b) $a = \frac{1}{3}$ $r = \frac{1}{3}$ $\frac{a}{1-r} = \frac{1/3}{1-1/3} = \frac{1}{2}$ //

**10.** (a) Find the first three terms in the binomial expansion of

$$\frac{2-x}{\sqrt{1+3x}}$$

in ascending powers of $x$.

**(5)**

(b) State the range of values of $x$ for which the expansion is valid.

**(1)**

(c) By writing $x = \frac{1}{22}$ in your expansion, find an approximate value for $\sqrt{22}$ in the form $\frac{a}{b}$, where $a$ and $b$ are integers to be found.

**(2)**

a) $(2-x)\left[1+3x\right]^{-\frac{1}{2}}$

$(2-x)\left[1+\frac{-1}{2}(3x)+\frac{(\frac{-1}{2})(\frac{-1}{2}-1)}{2}(3x)^2+\dots\right]$

$(2-x)\left[1+\frac{-3}{2}x+\frac{27}{8}x^2+\dots\right]$

$=\left[\begin{array}{l}2+-3x+\frac{27}{4}x^2+\dots \\ \quad -x+\frac{3}{2}x^2+\dots\end{array}\right]$

$= 2-4x+\frac{33}{4}x^2+\dots$

b) $-1 < 3x < 1$

$-\frac{1}{3} < x < \frac{1}{3}$

c) when $x = \frac{1}{22}$, $\frac{2-x}{\sqrt{1+3x}} = \frac{43/22}{\sqrt{25/22}} = \frac{43}{5\sqrt{22}}$

$= \frac{43\sqrt{22}}{22(5)}$

$\sqrt{22} = \left(2-4x+\frac{33}{4}x^2\right)\left(\frac{22(5)}{43}\right)$

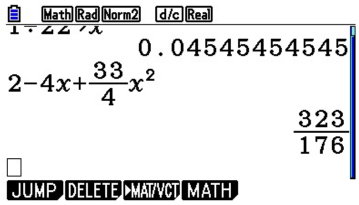
$\frac{323}{176}\cdot\frac{22\cdot 5}{43} = \frac{1615}{344}$ //

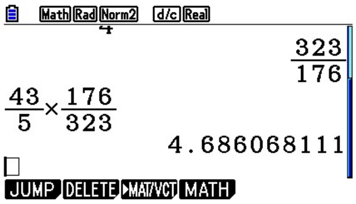
$$\frac{43}{5\sqrt{22}} \simeq 2-4x+\frac{33}{4}x^2$$

$$\frac{43}{5\sqrt{22}} \simeq \frac{323}{176}$$

$$\frac{43}{5}\frac{(176)}{(323)} = \sqrt{22}$$

$$\frac{7568}{1615} = \sqrt{22}$$



**11.** The variables $x$ and $y$ satisfy the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{1+x^2}$$

and $y = 2$ when $x = 0$.

Solve the differential equation, obtaining a simplified expression for $y$ in terms of $x$.

(7)

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{y}\,dy$$

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{y}\,dy$$

$$\frac{1}{2}\ln|1+x^2| = \ln y + C$$

$$0 = \ln 2 + C$$

$$C = -\ln 2$$

$$\ln\sqrt{1+x^2} = \ln\left(\frac{y}{2}\right)$$

$$y = 2\sqrt{1+x^2}$$

**12.** The curve $C$ is defined for $t \geq 0$ by the parametric equations

$$x = t^2 + t \qquad y = 4t^2 - t^3$$

A sketch of $C$ is shown in Figure 4 below.

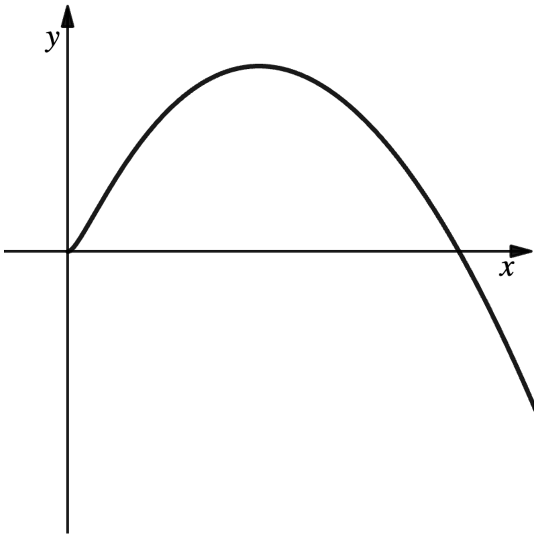



**Figure 3: https://www.desmos.com/calculator/qbyvqlaoh3**

(a) Find the gradient of $C$ at the point where it intersects the positive $x$-axis.

**(5)**

The area $A$ enclosed between $C$ and the $x$-axis is given by

$$A = \int_0^b y \, \mathrm{d}x$$

(b) State the value of $b$.

**(1)**

(c) Use parametric integration to show that

$$A = \int_0^c \left(4t^2 + 7t^3 - 2t^4\right) \mathrm{d}t$$

where $c$ is an integer to be found.

**(3)**

(d) Find the value of $A$.

**(1)**

**Question 12 continued**

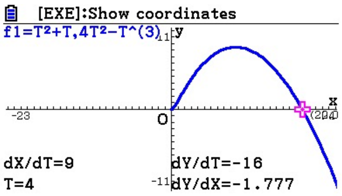
$y = 0$

$t^2[4 - t] = 0$

$t = 0$ or $t = 4$

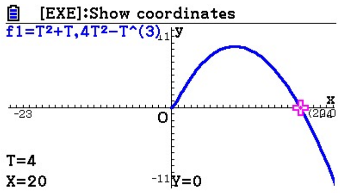
$\frac{dy}{dt} = 8t - t^2$

$\frac{dx}{dt} = 2t + 1$




$\frac{dy}{dx}\Big|_{t=4} = \frac{-16}{9}$ //

b) $\int_0^{20} y \, dx$




c) $\int_{x_0}^{x_1} y \, dx = \int_{t_0}^{t_1} y \frac{dx}{dt} \, dt$

$x = t^2 + t \qquad y = 4t^2 - t^3$

$x = 20 \Rightarrow t = 4$

$x = 0 \Rightarrow t = 0$

$\frac{dx}{dt} = 2t + 1$

$\int_0^4 (4t^2 - t^3)(2t + 1) \, dt$

$\int_0^4 8t^3 - 2t^4 + 4t^2 - t^3 \, dt$

$\int_0^4 4t^2 + 7t^3 - 2t^4 \, dt$

d) $\left[\frac{4t^3}{3} + \frac{7t^4}{4} - \frac{2t^5}{5}\right]_0^4 = \frac{1856}{15}$ //

**13.** A curve has equation

$$y = a\sin x + b\cos x$$

where $a$ and $b$ are constants.

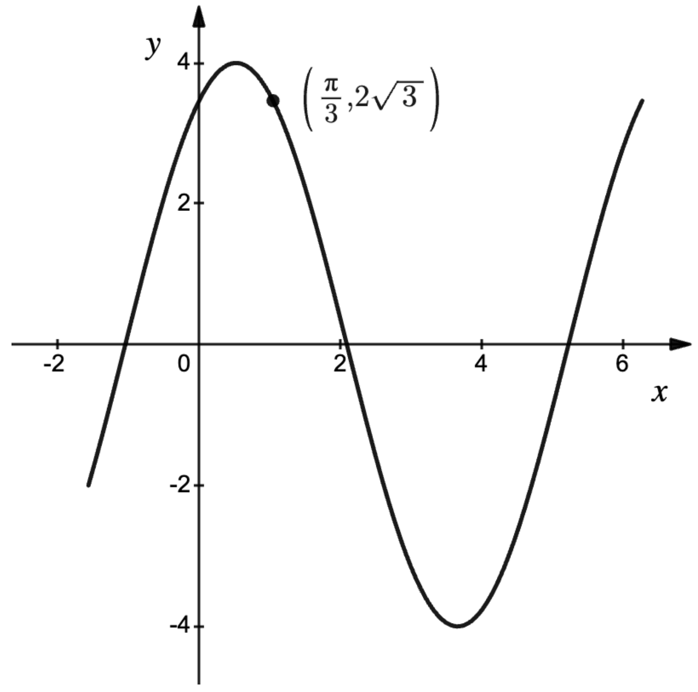


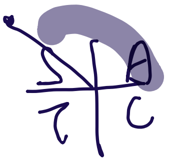

**Figure 4: https://www.desmos.com/calculator/yq7kymbzzj**

The maximum value of $y$ is 4 and the curve passes through the point $\left(\frac{\pi}{3}, 2\sqrt{3}\right)$, as shown in Figure 5.

Find the values of $a$ and $b$.

**(6)**

$R\sin(x+\alpha) \Rightarrow R=4$




$4\sin(x+\alpha) = 2\sqrt{3}$ when $x = \frac{\pi}{3}$

$x+\alpha = \pi - \frac{\pi}{3}$ since second point it is $2\sqrt{3}$.

$\frac{\pi}{3} + \alpha = \pi - \frac{\pi}{3}$ $\quad \alpha = \frac{1}{3}\pi$ //

$4\sin(x+\frac{\pi}{3}) = 4\sin x\cos\frac{\pi}{3} + 4\cos x\sin\frac{\pi}{3}$

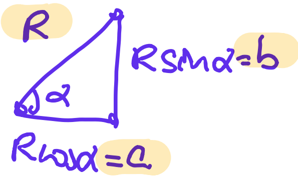
$= 2\sin x + 2\sqrt{3}\cos x$

$a=2$ and $b = 2\sqrt{3}$ //

$$y = a\sin x + b\cos x$$

$R\sin(x+\alpha) = \underbrace{R}\sin x\underbrace{\cos\alpha}_{a} + \underbrace{R}\cos x\underbrace{\sin\alpha}_{b}$

$R\cos\alpha = a$
$R\sin\alpha = b$




Sub in $x = \frac{\pi}{3}$, $y = 2\sqrt{3}$

$4\sqrt{3} = a\sqrt{3} + b$ ①

Also, max is 4 means, $\sqrt{a^2+b^2} = 4$

$a^2+b^2 = 16$

$b = \sqrt{3}(4-a)$ → $a^2 + [\sqrt{3}(4-a)]^2 = 16$

$a^2 + 3(4-a)^2 = 16$

$a^2 + 3(a^2 - 8a + 16) = 16$

$\vdots$

| | |
|---|---|
| $a = 4$ | $a = 2$ |
| $b = 0$ | $b = 2\sqrt{3}$ // |
| (rej) | |

**14.** (a) On Diagram 1 below, sketch on the same axes the graphs of $y = \frac{1}{x}$ and $y = \sec 2x$, hence show that the equation

$$\frac{1}{x} = \sec 2x$$

has exactly one solution for $x > 0$. (3)

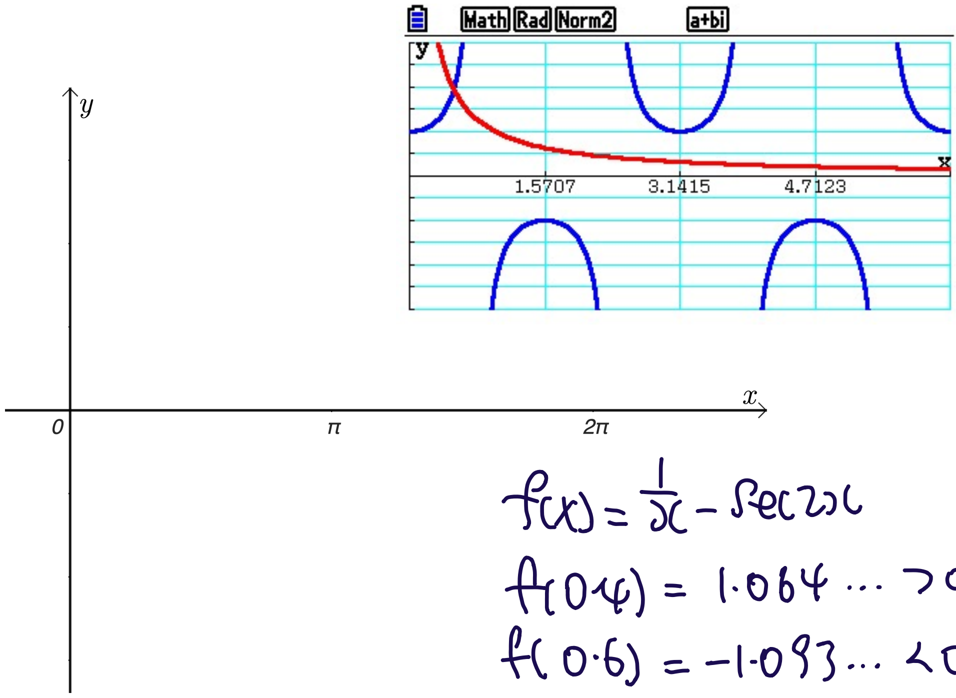



**Diagram 1**

$f(x) = \frac{1}{x} - \sec 2x$

$f(0.4) = 1.064\ldots > 0$

$f(0.6) = -1.093\ldots < 0$

change in sign, continuous over the interval,

(b) By considering a suitable change of sign, show that the solution to the equation lies between 0.4 and 0.6.

$0.4 < \alpha < 0.6$

(2)

(c) Show that the equation in part (a) can be rearranged to give

$$x = \frac{1}{2} \arccos x$$

$\frac{1}{x} = \frac{1}{\cos 2x}$

$x = \cos 2x$ (2)

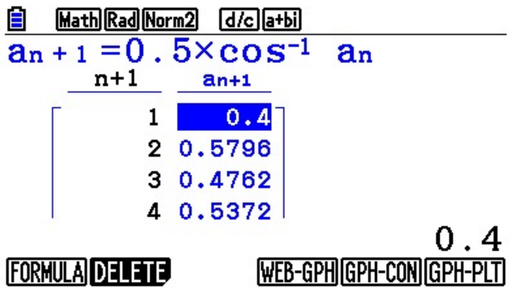
$\arccos x = 2x$

(d) Use the iterative formula

$$x_{n+1} = \frac{1}{2} \arccos x_n$$

$x = \frac{1}{2} \arccos x$

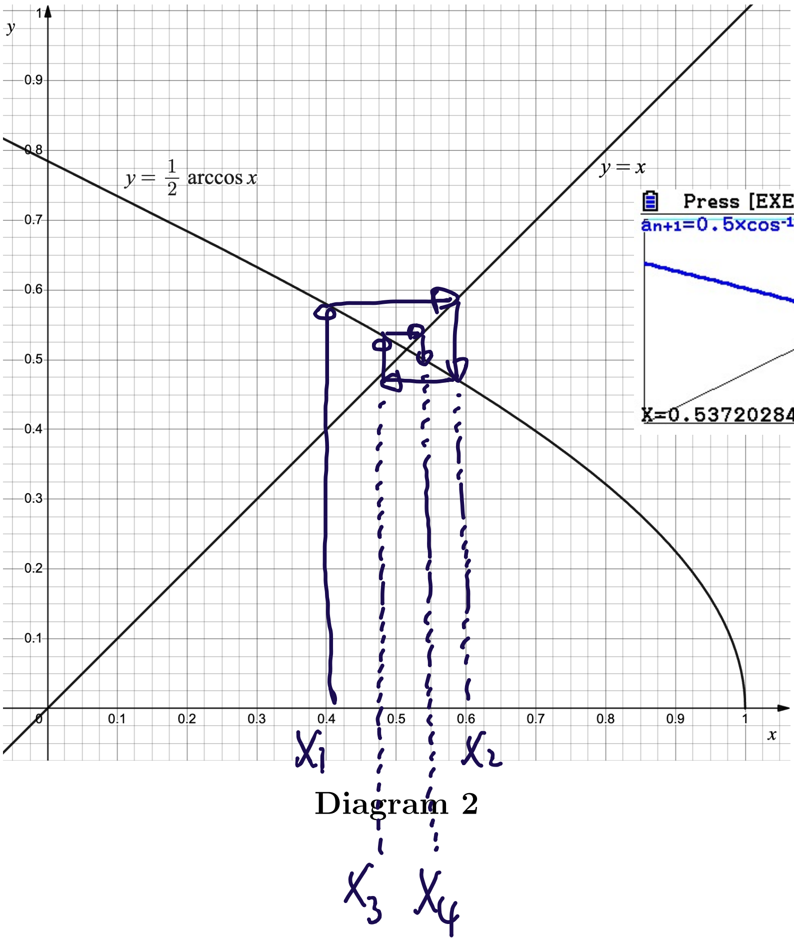
with $x_1 = 0.4$, to find $x_2$, $x_3$ and $x_4$, giving your answer to four decimal places. (2)



Question 14 continued

(e) On Diagram 2 below, draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of $x_2$, $x_3$ and $x_4$.

**(2)**



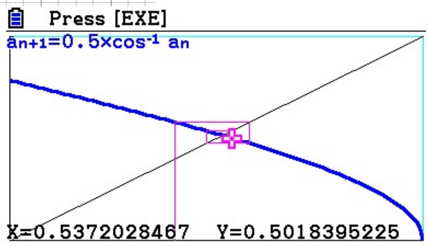




Diagram 2

**Question 14 continued**

Another copy of Diagram 1 if needed:

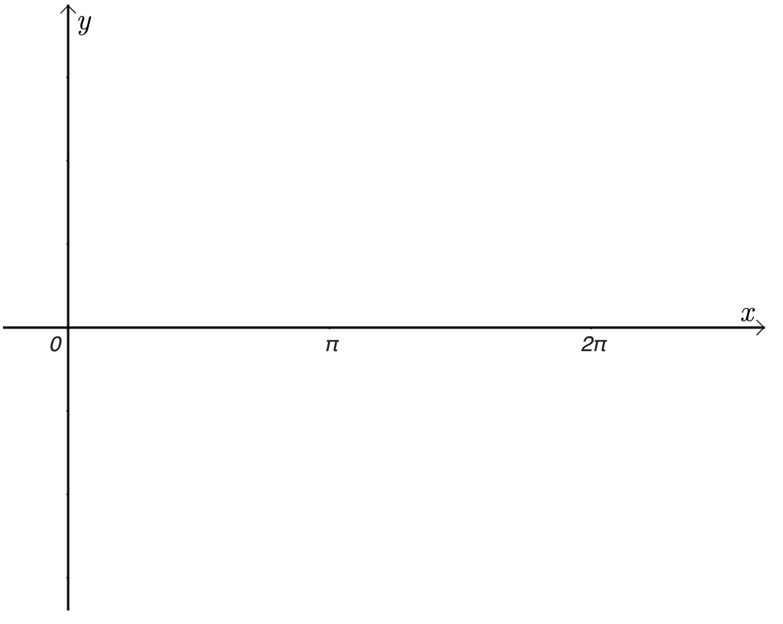



**Diagram 1**

Another copy of Diagram 2 if needed:

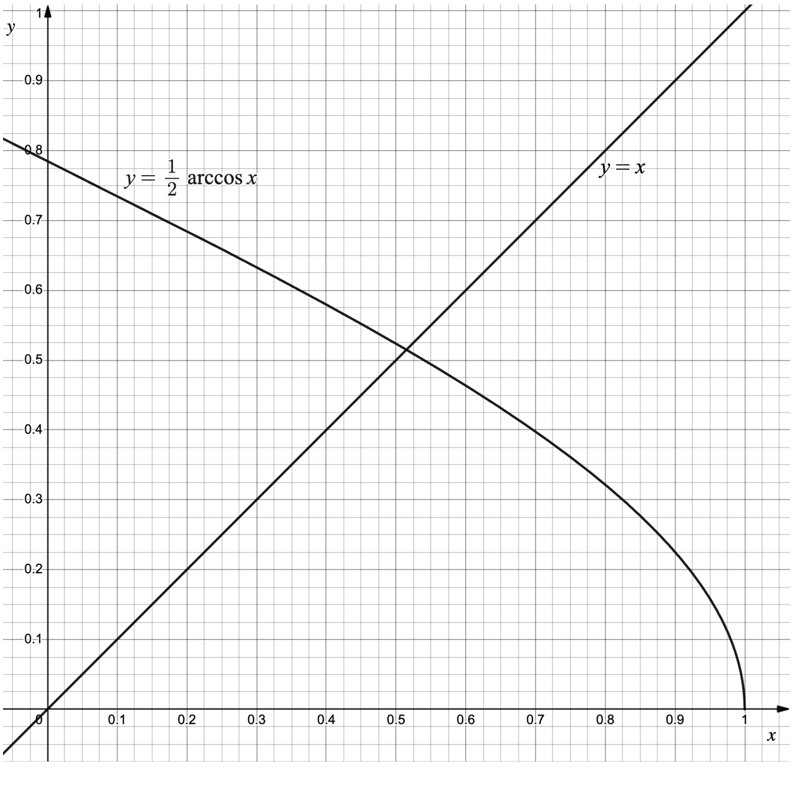
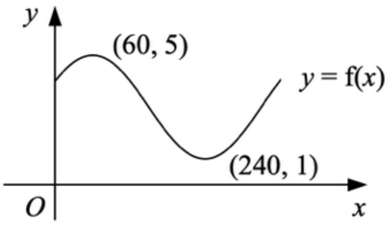



**Diagram 2**

**9**




The diagram shows the curve $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) \equiv a + b\sin x° + c\cos x°,\ x \in \mathbb{R},\ 0 \le x \le 360,$$

The curve has turning points with coordinates (60, 5) and (240, 1) as shown.

**a** State, with a reason, the value of the constant $a$. **(2)**

**b** Find the values of $k$ and $\alpha$, where $k > 0$ and $0 < \alpha < 90$, such that

$$\mathrm{f}(x) = a + k\sin(x + \alpha)°.$$ **(3)**

**c** Hence, or otherwise, find the exact values of the constants $b$ and $c$. **(3)**

**9** **a** $a = 3$

$b\sin x° + c\cos x°$ can be expressed in

the form $k\sin(x + \alpha)°$ which will vary

between $-k$ and $+k$

$\therefore\ a + k = 5$ and $a - k = 1$, hence $a = 3$

**b** $3 + k = 5\ \therefore\ k = 2$

$60 + \alpha = 90\ \therefore\ \alpha = 30$

**c** $\mathrm{f}(x) = 3 + 2\sin(x + 30)$

$= 3 + 2\sin x\cos 30 + 2\cos x\sin 30$

$= 3 + \sqrt{3}\sin x + \cos x$

$\therefore\ b = \sqrt{3},\ c = 1$

**(Total for Question 14 is 11 marks)**

**Total for paper is 100 marks**